

# QUIVER HECKE ALGEBRAS FOR ALTERNATING GROUPS

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**ABSTRACT.** The main result of this paper shows that, over large enough fields of characteristic different from 2, the alternating Hecke algebras are  $\mathbb{Z}$ -graded algebras that are isomorphic to fixed-point subalgebras of the quiver Hecke algebra of the symmetric group  $\mathfrak{S}_n$ . As a special case, this shows that the group algebra of the alternating group, over large enough fields of characteristic different from 2, is a  $\mathbb{Z}$ -graded algebra. We give a homogeneous presentation for these algebras, compute their graded dimension and show that the blocks of the quiver Hecke algebras of the alternating group are graded symmetric algebras.

## INTRODUCTION

In a landmark paper, Brundan and Kleshchev [2] constructed an explicit  $\mathbb{Z}$ -grading on the cyclotomic Hecke algebras of type  $A$ . These algebras include, as special cases, the group algebras of the symmetric group and the Iwahori-Hecke algebras of type  $A$ . This paper extends these results to the group algebras of the alternating groups and, more generally, to Mitsuhashi's alternating Hecke algebras [20].

Let  $\mathcal{H}_\xi(\mathfrak{S}_n)$  be the Iwahori-Hecke algebra of the symmetric group with parameter  $\xi \in F^\times$ , where  $F$  is a field. Then  $\mathcal{H}_\xi(\mathfrak{S}_n)$  is a deformation of the group algebra of  $\mathfrak{S}_n$ . The algebra  $\mathcal{H}_\xi(\mathfrak{S}_n)$  has an automorphism  $\#$  that can be considered as a  $\xi$ -deformation of the sign automorphism of  $F\mathfrak{S}_n$ . The alternating Hecke algebra  $\mathcal{H}_\xi(\mathfrak{A}_n) = \mathcal{H}_\xi(\mathfrak{S}_n)^\#$  is the fixed-point subalgebra of  $\mathcal{H}_\xi(\mathfrak{S}_n)$  under  $\#$ .

Brundan and Kleshchev showed that  $\mathcal{H}_\xi(\mathfrak{S}_n)$  is a  $\mathbb{Z}$ -graded algebra by constructing an explicit family of isomorphisms  $\theta : \mathcal{R}_e(\mathfrak{S}_n) \xrightarrow{\sim} \mathcal{H}_\xi(\mathfrak{S}_n)$ , where  $\mathcal{R}_e(\mathfrak{S}_n)$  is a quiver Hecke algebra of  $\mathfrak{S}_n$  [2, 5, 14, 22]. Here,  $e$  is **quantum characteristic** of  $\xi$ , so  $e > 0$  is minimal such that  $1 + \xi + \cdots + \xi^{e-1} = 0$ .

As observed in [15, (3.14)], the algebra  $\mathcal{R}_e(\mathfrak{S}_n)$  has a homogeneous automorphism **sgn** that is a graded analogue of the sign automorphism of the symmetric group. Let  $\mathcal{R}_e(\mathfrak{A}_n) = \mathcal{R}_e(\mathfrak{S}_n)^{\text{sgn}}$  be the fixed-point subalgebra of  $\mathcal{R}_e(\mathfrak{S}_n)$  under **sgn**. Then  $\mathcal{R}_e(\mathfrak{A}_n)$  is a homogeneous subalgebra of  $\mathcal{R}_e(\mathfrak{S}_n)$ . It is natural to hope that  $\theta$  restricts to an isomorphism  $\mathcal{R}_e(\mathfrak{A}_n) \xrightarrow{\sim} \mathcal{H}_\xi(\mathfrak{A}_n)$ . Unfortunately, the isomorphisms constructed by Brundan and Kleshchev do not restrict to isomorphisms between the alternating subalgebras; see [Example 2.20](#).

Let  $F$  be a field and  $\xi \in F$  an element of quantum characteristic  $e$ . By **definition**, the field  $F$  is **large enough** for  $\xi$  if  $F$  contains square roots  $\sqrt{\xi}$  and  $\sqrt{1 + \xi + \xi^2}$  whenever  $e > 3$ .

2000 *Mathematics Subject Classification.* 20C08, 20D06, 20C30.

*Key words and phrases.* Alternating groups, alternating Hecke algebras, Khovanov-Lauda-Rouquier algebras, representation theory.

**Theorem A.** *Suppose that  $\xi \in F$  is an element of quantum characteristic  $e \neq 2$ , where  $F$  is a large enough field for  $\xi$  of characteristic different from 2. Then  $\mathcal{H}_\xi(\mathfrak{A}_n) \cong \mathcal{R}_e(\mathfrak{A}_n)$ .*

To prove this result we construct a new isomorphism  $\mathcal{R}_e(\mathfrak{S}_n) \xrightarrow{\sim} \mathcal{H}_\xi(\mathfrak{S}_n)$  that intertwines the involutions, **sgn** and **#**, on the two algebras. We do this using the framework developed by Hu and the second-named author [9], which shows that the KLR grading can be described explicitly in terms of seminormal forms.

As the algebra  $\mathcal{R}_e(\mathfrak{A}_n)$  is a graded subalgebra of  $\mathcal{R}_e(\mathfrak{S}_n)$  we immediately obtain the following.

**Corollary A1.** *Suppose that  $\xi \in F$  is an element of quantum characteristic  $e \neq 2$ , where  $F$  is a large enough field for  $\xi$  of characteristic different from 2. Then  $\mathcal{H}_\xi(\mathfrak{A}_n)$  is a  $\mathbb{Z}$ -graded algebra.*

In particular, over large enough fields of characteristic different from 2 the group algebra  $F\mathfrak{A}_n$  of the alternating group is  $\mathbb{Z}$ -graded, for  $n \geq 1$ . The alternating group corresponds to the case when  $\xi = 1$ , so if  $F$  is a field of characteristic  $p \neq 2$  then  $F\mathfrak{A}_n \cong \mathcal{R}_p(\mathfrak{A}_n)$  if 3 has a square root in  $F$  whenever  $p > 3$ .

Applying [Theorem A](#) twice shows that, up to isomorphism,  $\mathcal{H}_\xi(\mathfrak{A}_n)$  depends only on  $e$ , the quantum characteristic of  $\xi$ , rather than on  $\xi$  itself. Hence, the following holds:

**Corollary A2.** *Let  $F$  be a field of characteristic different from 2. Suppose that  $\xi, \xi' \in F$  are elements of quantum characteristic  $e \neq 2$ , where  $F$  is a large enough field for  $\xi$  and for  $\xi'$ . Then  $\mathcal{H}_\xi(\mathfrak{A}_n) \cong \mathcal{H}_{\xi'}(\mathfrak{A}_n)$ .*

In particular, over a large enough field, the decomposition matrix of  $\mathcal{H}_\xi(\mathfrak{A}_n)$  depends only on  $e$ , and the field  $F$ , and not on the choice of  $\xi$ .

The quiver Hecke algebra  $\mathcal{R}_e(\mathfrak{S}_n)$  has a homogeneous presentation by generators and relations that is described in terms of the quiver  $\Gamma_e$  with vertex set  $I = \mathbb{Z}/e\mathbb{Z}$  and edges  $i \rightarrow i+1$ , for  $i \in I$ . In [Section 1.3](#) we associate to  $\Gamma_e$  a set of simple roots  $\{\alpha_i \mid i \in I\}$  and a positive root lattice  $Q^+ = \bigoplus_{i \in I} \mathbb{N}\alpha_i$ . If  $\alpha \in Q^+$  let  $I^\alpha = \{\mathbf{i} \in I^n \mid \alpha = \alpha_{i_1} + \dots + \alpha_{i_n}\}$ . Let  $\sim$  be the equivalence relation on  $I^n$  generated by  $\mathbf{i} \sim \mathbf{j}$  if  $\mathbf{j} = -\mathbf{i}$ . This relation induces an equivalence relation on  $Q^+$  where  $\alpha \sim \beta$  if there exists  $\mathbf{i} \in I^\alpha$  such that  $-\mathbf{i} \in I^\beta$ . Let  $Q_n^\varepsilon = Q_n^+ / \sim$ . If  $\gamma \in Q_n^\varepsilon$  let  $I^\gamma = \bigcup_{\alpha \in \gamma} I^\alpha$ .

Using the KLR presentation of  $\mathcal{R}_e(\mathfrak{S}_n)$ , and the realisation of  $\mathcal{R}_e(\mathfrak{A}_n)$  as a fixed-point subalgebra of  $\mathcal{R}_e(\mathfrak{S}_n)$ , gives the following homogeneous presentation for  $\mathcal{R}_e(\mathfrak{A}_n)$  with respect to the  $\mathbb{Z}$ -grading. For  $\mathbf{i}, \mathbf{j} \in I^n$  set  $\delta_{\mathbf{i} \sim \mathbf{j}} = 1$  if  $\mathbf{i} \sim \mathbf{j}$  and set  $\delta_{\mathbf{i} \not\sim \mathbf{j}} = 0$  otherwise.

**Theorem B.** *Suppose that  $e \neq 2$  and that 2 is invertible in  $\mathbb{Z}$ . Then*

$$\mathcal{R}_e(\mathfrak{A}_n) = \bigoplus_{\gamma \in Q_n^\varepsilon} \mathcal{R}_e(\mathfrak{A}_n)_\gamma$$

where  $\mathcal{R}_e(\mathfrak{A}_n)_\gamma$  is the unital associative  $\mathbb{Z}$ -algebra generated by elements

$$\{\Psi_r(\mathbf{i}), Y_s(\mathbf{i}), \varepsilon(\mathbf{i}) \mid 1 \leq r \leq n, 1 \leq s < n \text{ and } \mathbf{i} \in I^\gamma\}$$

subject to the relations

$$\begin{aligned}
 Y_1(\mathbf{i})^{(\Lambda_0, \alpha_{i_1})} &= 0, & \varepsilon(\mathbf{i})\varepsilon(\mathbf{j}) &= \delta_{\mathbf{i} \sim \mathbf{j}} \varepsilon(\mathbf{i}), & \sum_{\mathbf{i} \in I^\gamma} \frac{1}{2} \varepsilon(\mathbf{i}) &= 1, \\
 Y_r(-\mathbf{i}) &= -Y_r(\mathbf{i}) & \Psi_r(-\mathbf{i}) &= -\Psi_r(\mathbf{i}) & \varepsilon(-\mathbf{i}) &= \varepsilon(\mathbf{i}) \\
 \varepsilon(\mathbf{i})Y_r(\mathbf{j})\varepsilon(\mathbf{k}) &= \delta_{\mathbf{ij}}\delta_{\mathbf{jk}}Y_r(\mathbf{j}), & \varepsilon(\mathbf{i})\Psi_r(\mathbf{j})\varepsilon(\mathbf{k}) &= \delta_{s_r \cdot \mathbf{i}, \mathbf{j}}\delta_{\mathbf{jk}}\Psi_r(\mathbf{j}), & Y_r(\mathbf{i})Y_s(\mathbf{j}) &= \delta_{\mathbf{ij}}Y_s(\mathbf{i})Y_r(\mathbf{j}), \\
 \Psi_r(\mathbf{i})Y_{r+1}(\mathbf{i}) &= (Y_r(s_r \cdot \mathbf{i})\Psi_r(\mathbf{i}) + \delta_{i_r, i_{r+1}}\varepsilon(\mathbf{i})), \\
 Y_{r+1}(s_r \cdot \mathbf{i})\Psi_r(\mathbf{i}) &= (\Psi_r(\mathbf{i})Y_r(\mathbf{i}) + \delta_{i_r, i_{r+1}}\varepsilon(\mathbf{i})), \\
 \Psi_r(\mathbf{i})Y_s(\mathbf{i}) &= Y_s(s_r \cdot \mathbf{i})\Psi_r(\mathbf{i}), & \text{if } s \neq r, r+1, \\
 \Psi_r(s_t \cdot \mathbf{i})\Psi_t(\mathbf{i}) &= \Psi_t(s_r \cdot \mathbf{i})\Psi_r(\mathbf{i}), & \text{if } |r-s| > 1, \\
 \Psi_r(s_r \cdot \mathbf{i})\Psi_r(\mathbf{i}) &= \begin{cases} Y_r(\mathbf{i}) - Y_{r+1}(\mathbf{i}), & \text{if } i_r \rightarrow i_{r+1}, \\ Y_{r+1}(\mathbf{i}) - Y_r(\mathbf{i}), & \text{if } i_r \leftarrow i_{r+1}, \\ 0, & \text{if } i_r = i_{r+1}, \\ \varepsilon(\mathbf{i}), & \text{otherwise} \end{cases}
 \end{aligned}$$

and  $\Psi_r(s_{r+1}s_r \cdot \mathbf{i})\Psi_{r+1}(s_r \cdot \mathbf{i})\Psi_r(\mathbf{i}) - \Psi_{r+1}(s_r s_{r+1} \cdot \mathbf{i})\Psi_r(s_{r+1} \cdot \mathbf{i})\Psi_{r+1}(\mathbf{i})$  is equal to

$$\begin{cases} \varepsilon(\mathbf{i}), & \text{if } i_r = i_{r+2} \leftarrow i_{r+1}, \\ -\varepsilon(\mathbf{i}), & \text{if } i_r = i_{r+2} \rightarrow i_{r+1}, \\ 0, & \text{otherwise,} \end{cases}$$

for all  $\mathbf{i}, \mathbf{j}, \mathbf{k} \in I^\gamma$  and all admissible  $r, s$  and  $t$ .

The fraction  $\frac{1}{2}$  appears in the relation  $\sum_{\mathbf{i} \in I^\gamma} \frac{1}{2} \varepsilon(\mathbf{i}) = 1$  because  $\varepsilon(\mathbf{i}) = \varepsilon(-\mathbf{i})$ . (It follows from the relations in [Theorem B](#) that  $\varepsilon(\mathbf{i}) = 0$  if  $\mathbf{i} = -\mathbf{i}$ , for  $\mathbf{i} \in I^n$ .)

The relations in [Theorem B](#) are homogeneous with respect to the degree function

$$\deg \varepsilon(\mathbf{i}) = 0, \quad \deg Y_r(\mathbf{i}) = 2 \quad \text{and} \quad \deg \Psi_r(\mathbf{i}) = -(\alpha_{i_r}, \alpha_{i_{r+1}}),$$

where  $(\cdot, \cdot)$  is the Cartan pairing. Hence,  $\mathcal{R}_e(\mathfrak{A}_n)$  is a  $\mathbb{Z}$ -graded algebra. Quite surprisingly, this presentation is almost identical to the KLR-presentation of  $\mathcal{R}_e(\mathfrak{S}_n)$ . The main difference being the three “sign relations” relating the generators indexed by  $\mathbf{i}$  and  $-\mathbf{i}$ , for  $\mathbf{i} \in I^\gamma$ . The key idea behind the proof [Theorem B](#) is to introduce a  $(\mathbb{Z}_2 \times \mathbb{Z})$ -grading on the KLR algebra  $\mathcal{R}_e(\mathfrak{S}_n)$ . With respect to the  $(\mathbb{Z}_2 \times \mathbb{Z})$ -grading, the algebra  $\mathcal{R}_e(\mathfrak{A}_n)$  is the even part of  $\mathcal{R}_e(\mathfrak{S}_n)$ . Using this observation we can deduce the relations for  $\mathcal{R}_e(\mathfrak{A}_n)$  directly from those for  $\mathcal{R}_e(\mathfrak{S}_n)$ .

Finally, using the graded cellular bases of Hu and Mathas [\[7\]](#) we construct a homogeneous basis for the  $\mathbb{Z}$ -graded algebra  $\mathcal{R}_e(\mathfrak{A}_n)$ . As a corollary we obtain the graded dimension of  $\mathcal{R}_e(\mathfrak{A}_n)$ . See [Chapter 3](#) below for the unexplained notation.

**Theorem C.** *Suppose that  $e \neq 2$  and that 2 is invertible in  $F$ . Then the alternating quiver Hecke algebra  $\mathcal{R}_e(\mathfrak{A}_n)$  has graded dimension*

$$\sum_{\substack{(\mathbf{s}, \mathbf{t}) \in \text{Std}^2(\mathcal{P}_n) \\ \text{res}(\mathbf{s}) \in I_+^n}} q^{\deg \mathbf{s} + \deg \mathbf{t}}.$$

In fact, using the work of Li [\[16\]](#) it follows that over any ring in which 2 is invertible the algebra  $\mathcal{R}_e(\mathfrak{A}_n)$  is free with the same graded rank. As a second application of our basis theorem we show that the blocks of  $\mathcal{R}_e(\mathfrak{A}_n)$  are graded symmetric algebras.

The results in this paper exclude the cases when  $F$  is a field of characteristic 2 and when  $e = 2$  or, equivalently,  $\xi = -1$ . This is because most of our arguments fail, and most of our results are false, when we drop these assumptions.

The paper is organised as follows. [Chapter 1](#) starts by defining the quiver Hecke algebra  $\mathcal{R}_e(\mathfrak{A}_n)$  of the alternating group as the fixed-point subalgebra of  $\mathcal{R}_e(\mathfrak{S}_n)$  under the homogeneous sign involution  $\text{sgn}$ . We then prove [Theorem B](#) by first introducing a  $(\mathbb{Z}_2 \times \mathbb{Z})$ -grading on  $\mathcal{R}_e(\mathfrak{S}_n)$  and showing that  $\mathcal{R}_e(\mathfrak{A}_n)$  is the *even* part of  $\mathcal{R}_e(\mathfrak{S}_n)$ , with respect to the  $\mathbb{Z}_2$ -grading. [Chapter 2](#) starts by setting up the framework of *seminormal coefficient systems* and showing how seminormal bases behave under the ungraded sign involution  $\#$ . Building on ideas from [9], we give a new presentation of  $\mathcal{H}_t^\mathcal{O}$ , over a specially chosen ring  $\mathcal{O}$ , that we use to construct a new isomorphism  $\Theta : \mathcal{R}_e(\mathfrak{S}_n) \xrightarrow{\sim} \mathcal{H}_\xi(\mathfrak{S}_n)$  over the residue field of  $\mathcal{O}$ . Unlike the known isomorphisms in the literature,  $\Theta$  intertwines the two sign involutions,  $\text{sgn}$  and  $\#$ , implying [Theorem A](#). In [Chapter 3](#) we give a homogeneous basis of  $\mathcal{R}_e(\mathfrak{A}_n)$  and hence prove [Theorem C](#). As an application we show that the blocks of  $\mathcal{R}_e(\mathfrak{A}_n)$  are graded symmetric algebras. Finally, using Clifford theory, the classification of the blocks and irreducible graded modules of  $\mathcal{R}_e(\mathfrak{A}_n)$  is given.

**Acknowledgments.** We thank the referee for their careful reading of our manuscript. The first-named author was supported by an Australian Postgraduate Award and the second-named author by the Australian Research Council. Some of the work in the first-named author's PhD thesis [1] was based on an earlier version of this paper.

## 1. IWAHORI-HECKE ALGEBRAS AND QUIVER HECKE ALGEBRAS

This chapter defines both the alternating Hecke algebras and the alternating quiver Hecke algebras of type A. Both algebras are defined as fixed point subalgebras of the corresponding Hecke algebras. In the final section we prove that [Theorem B](#) gives a homogeneous presentation for the alternating quiver Hecke algebra.

**1.1. Iwahori-Hecke algebras and alternating Hecke algebras.** We start by defining the Iwahori-Hecke algebras of the symmetric groups. These algebras are well-studied deformations of the group algebras of the symmetric groups that arise naturally in the representation theory of the general linear groups.

Fix a (unital) integral domain  $\mathcal{Z}$  and an invertible element  $\xi \in \mathcal{Z}^\times$ .

**1.1. Definition.** The **Iwahori-Hecke algebra**  $\mathcal{H}_\xi(\mathfrak{S}_n) = \mathcal{H}_\xi^\mathcal{Z}(\mathfrak{S}_n)$  is the (unital) associative  $\mathcal{Z}$ -algebra with generators  $T_1, \dots, T_{n-1}$  subject to relations

$$\begin{aligned} (T_r - \xi)(T_r + 1) &= 0, & \text{for } r = 1, \dots, n-1, \\ T_r T_s &= T_s T_r, & \text{if } |r - s| > 1, \\ T_r T_{r+1} T_r &= T_{r+1} T_r T_{r+1}, & \text{for } r = 1, \dots, n-2. \end{aligned}$$

For  $1 \leq r < n$  let  $s_r = (r, r+1) \in \mathfrak{S}_n$ . Then  $\{s_1, \dots, s_{n-1}\}$  is the standard set of Coxeter generators for  $\mathfrak{S}_n$ . If  $w \in \mathfrak{S}_n$  then the **length** of  $w$  is the integer  $\ell(w) = \min\{l \geq 0 \mid w = s_{r_1} \dots s_{r_l} \text{ with } 1 \leq r_j < n\}$ . A **reduced expression** for  $w$  is any word  $w = s_{r_1} \dots s_{r_\ell}$  with  $\ell = \ell(w)$  and  $1 \leq r_j < n$ , for  $1 \leq j \leq \ell$ .

If  $w \in \mathfrak{S}_n$  define  $T_w = T_{r_1} \dots T_{r_\ell}$ , where  $w = s_{r_1} \dots s_{r_\ell}$  is any reduced expression. As is well-known, because the braid relations hold in  $\mathcal{H}_\xi(\mathfrak{S}_n)$  the element  $T_w$

depends only on  $w$  and not on the choice of reduced expression. Moreover, the algebra  $\mathcal{H}_\xi(\mathfrak{S}_n)$  is free as a  $\mathcal{Z}$ -module with basis  $\{T_w \mid w \in \mathfrak{S}_n\}$ . See, for example, [17, Chapter 1]. In particular, if  $\xi = 1$  then  $\mathcal{H}_\xi(\mathfrak{S}_n) \cong \mathcal{Z}\mathfrak{S}_n$  via the map  $T_w \mapsto w$ , for  $w \in \mathfrak{S}_n$ .

Following Goldman [10, Theorem 5.4], let  $\# : \mathcal{H}_\xi(\mathfrak{S}_n) \rightarrow \mathcal{H}_\xi(\mathfrak{S}_n)$  be the unique  $\mathcal{Z}$ -linear automorphism of  $\mathcal{H}_\xi(\mathfrak{S}_n)$  such that

$$(1.2) \quad T_r^\# = -\xi T_r^{-1} = -T_r + (\xi - 1),$$

for  $1 \leq r < n$ . It follows directly from the definitions that when  $\xi = 1$  the automorphism  $\#$  of  $\mathcal{H}_1^\mathcal{Z}(\mathfrak{S}_n) \cong \mathcal{Z}\mathfrak{S}_n$  is the usual “sign involution” which sends each simple transposition  $s_r$  to  $-s_r$ , for  $1 \leq r < n$  [11, p5]. Since the group algebra of the alternating group is the fixed-point subalgebra of the sign automorphism, the following definition gives a  $\xi$ -analogue of the group ring  $\mathcal{Z}\mathfrak{A}_n$  of the alternating group  $\mathfrak{A}_n$ .

**1.3. Definition** (Mitsuhashi [20]). Suppose that  $\xi \neq -1$ . Then the **alternating Hecke algebra** is the fixed-point subalgebra

$$\mathcal{H}_\xi^\#(\mathfrak{A}_n) = \mathcal{H}_\xi^\mathcal{Z}(\mathfrak{A}_n) = \{h \in \mathcal{H}_\xi(\mathfrak{S}_n) \mid h^\# = h\}$$

of  $\mathcal{H}_\xi(\mathfrak{S}_n)$  under the hash involution.

**1.4. Remark.** Mitsuhashi’s [20, Definition 4.1] original definition of the alternating Hecke algebra was by generators and relations, giving a deformation of a well-known presentation of the alternating group. Definition 1.3 is equivalent to Mitsuhashi’s definition by [19, Proposition 1.5].

**1.2. Graded modules and algebras.** The main result of this paper shows that the alternating Hecke algebra is a graded algebra, so we quickly review this terminology. For the most part we will work with  $\mathbb{Z}$ -graded modules and algebras, however, to prove Theorem B we consider more general gradings.

Recall that  $\mathcal{Z}$  is a unital integral domain. In this paper all modules will be assumed to be free and of finite rank as  $\mathcal{Z}$ -modules.

Let  $(G, +)$  be an abelian group. A  **$G$ -graded  $\mathcal{Z}$ -module** is a  $\mathcal{Z}$ -module  $M$  that admits a vector space decomposition  $M = \bigoplus_{g \in G} M_g$ . If  $g \in G$  and  $0 \neq m \in M_g$  then  $m$  is **homogeneous of degree  $g$** . Similarly, a  **$G$ -graded  $\mathcal{Z}$ -algebra** is a  $G$ -graded  $\mathcal{Z}$ -module  $A = \bigoplus_{g \in G} A_g$  that is a  $\mathcal{Z}$ -algebra such that  $A_f A_g \subseteq A_{f+g}$ , for all  $f, g \in G$ . A  **$G$ -graded  $A$ -module** is a  $G$ -graded  $\mathcal{Z}$ -module  $M$  such that  $M_f A_g \subseteq M_{f+g}$ , for  $f, g \in G$ .

Unless otherwise specified,  $G = \mathbb{Z}$  and a **graded module** will mean a  $\mathbb{Z}$ -graded module. Similarly, a **graded algebra** is a  $\mathbb{Z}$ -graded algebra.

**1.3. Cyclotomic quiver Hecke algebras of type  $A$ .** We now define the second class of algebras that we are interested in: the cyclotomic quiver Hecke algebras of  $\mathfrak{S}_n$ . These algebras are certain quotients of the  $\mathbb{Z}$ -graded quiver Hecke algebras introduced, independently, by Khovanov and Lauda [14] and Rouquier [22].

Fix  $e \in \{3, 4, 5, \dots\} \cup \{\infty\}$  and define  $\Gamma_e$  to be the quiver with vertex set  $I = \mathbb{Z}/e\mathbb{Z}$  and edges  $i \rightarrow i+1$ , for  $i \in I$ . (By convention,  $I = \mathbb{Z}$  if  $e = \infty$ .) Thus,  $\Gamma_e$  is the infinite quiver of type  $A_\infty$  if  $e = \infty$  and the finite quiver of Dynkin type  $A_{e-1}^{(1)}$  if  $e \geq 3$ . We exclude  $e = 2$  only because this corresponds to the case  $\xi = -1$  in Definition 1.3, which we do not consider in this paper.

Following Kac [13], to the quiver  $\Gamma_e$  we attach the usual Lie theoretic data of the positive roots  $\{\alpha_i \mid i \in I\}$ , the fundamental weights  $\{\Lambda_i \mid i \in I\}$ , the positive weight lattice  $P^+ = \bigoplus_{i \in I} \mathbb{N}\Lambda_i$ , the positive root lattice  $Q^+ = \bigoplus_{i \in I} \mathbb{N}\alpha_i$ , the non-degenerate pairing  $(\ , \ ) : P^+ \times Q^+ \rightarrow \mathbb{Z}$  given by  $(\Lambda_i, \alpha_j) = \delta_{ij}$ , for  $i, j \in I$ , and the **Cartan matrix**  $C = (c_{ij})_{i,j \in I}$  where

$$c_{ij} = \begin{cases} 2, & \text{if } i = j, \\ -1, & \text{if } i \leftarrow j \text{ or } i \rightarrow j, \\ 0, & \text{otherwise.} \end{cases}$$

The **height** of  $\alpha \in Q^+$  is the non-negative integer  $\text{ht } \alpha = \sum_i (\Lambda_i, \alpha)$ . Fix  $n \geq 0$  and let  $Q_n^+ = \{\alpha \in Q^+ \mid \text{ht } \alpha = n\}$ . For  $\alpha \in Q_n^+$ , let

$$I^\alpha = \{\mathbf{i} = (i_1, \dots, i_n) \in I^n \mid \alpha = \alpha_{i_1} + \dots + \alpha_{i_n}\}.$$

**1.5. Definition** (Khovanov and Lauda [14] and Rouquier [22]). Suppose that  $\alpha \in Q^+$  and  $e \in \{3, 4, 5, \dots\} \cup \{\infty\}$ . The **cyclotomic quiver Hecke algebra**  $\mathcal{R}_e(\mathfrak{S}_n)_\alpha$  is the unital associative  $\mathbb{Z}$ -algebra with generators

$$\{\psi_1, \dots, \psi_{n-1}\} \cup \{y_1, \dots, y_n\} \cup \{e(\mathbf{i}) \mid \mathbf{i} \in I^\alpha\}$$

and relations

$$\begin{aligned} y_1^{(\Lambda_0, \alpha_{i_1})} e(\mathbf{i}) &= 0, & e(\mathbf{i}) e(\mathbf{j}) &= \delta_{\mathbf{ij}} e(\mathbf{i}), & \sum_{\mathbf{i} \in I^\alpha} e(\mathbf{i}) &= 1, \\ y_r e(\mathbf{i}) &= e(\mathbf{i}) y_r, & \psi_r e(\mathbf{i}) &= e(s_r \cdot \mathbf{i}) \psi_r, & y_r y_s &= y_s y_r, \\ \psi_r y_{r+1} e(\mathbf{i}) &= (y_r \psi_r + \delta_{i_r, i_{r+1}}) e(\mathbf{i}), & y_{r+1} \psi_r e(\mathbf{i}) &= (\psi_r y_r + \delta_{i_r, i_{r+1}}) e(\mathbf{i}), \\ \psi_r y_s &= y_s \psi_r, & & \text{if } s \neq r, r+1, \\ \psi_r \psi_s &= \psi_s \psi_r, & & \text{if } |r-s| > 1, \\ \psi_r^2 e(\mathbf{i}) &= \begin{cases} 0, & \text{if } i_r = i_{r+1}, \\ (y_r - y_{r+1}) e(\mathbf{i}), & \text{if } i_r \rightarrow i_{r+1}, \\ (y_{r+1} - y_r) e(\mathbf{i}), & \text{if } i_r \leftarrow i_{r+1}, \\ e(\mathbf{i}), & \text{otherwise,} \end{cases} \\ \psi_r \psi_{r+1} \psi_r e(\mathbf{i}) &= \begin{cases} (\psi_{r+1} \psi_r \psi_{r+1} - 1) e(\mathbf{i}), & \text{if } i_r = i_{r+2} \rightarrow i_{r+1}, \\ (\psi_{r+1} \psi_r \psi_{r+1} + 1) e(\mathbf{i}), & \text{if } i_r = i_{r+2} \leftarrow i_{r+1}, \\ \psi_{r+1} \psi_r \psi_{r+1} e(\mathbf{i}), & \text{otherwise,} \end{cases} \end{aligned}$$

for  $\mathbf{i}, \mathbf{j} \in I^\alpha$  and all admissible  $r$  and  $s$ . If  $n \geq 0$  then the **quiver Hecke algebra of  $\mathfrak{S}_n$**  is the algebra

$$\mathcal{R}_e(\mathfrak{S}_n) = \bigoplus_{\alpha \in Q_n^+} \mathcal{R}_e(\mathfrak{S}_n)_\alpha.$$

Note that the algebra  $\mathcal{R}_e(\mathfrak{S}_n)_\alpha$  depends on  $e, \Gamma_e, \Lambda_0$  and  $\alpha \in Q^+$ .

We write  $\mathcal{R}_e(\mathfrak{S}_n) = \mathcal{R}_e^{\mathbb{Z}}(\mathfrak{S}_n)$  when we want to emphasise that  $\mathcal{R}_e(\mathfrak{S}_n)$  is a  $\mathbb{Z}$ -algebra. The main advantage of the relations in Definition 1.5 is that they are homogeneous with respect to the following  $\mathbb{Z}$ -valued degree function:

$$(1.6) \quad \begin{aligned} \deg e(\mathbf{i}) &= 0, & \text{for all } \mathbf{i} \in I^n, \\ \deg y_r &= 2, & \text{for } 1 \leq r \leq n, \\ \deg \psi_r e(\mathbf{i}) &= -c_{i_r, i_{r+1}}, & \text{for } 1 \leq r < n \text{ and } \mathbf{i} \in I^n. \end{aligned}$$

Therefore,  $\mathcal{R}_e(\mathfrak{S}_n)$  is a  $\mathbb{Z}$ -graded algebra.

1.7. *Remark.* There are fewer relations appearing in [Definition 1.5](#) than in [\[2, Theorem 1.1\]](#). This is because we are assuming that  $e \neq 2$  (and  $\Lambda = \Lambda_0$ ). We have also made a sign change compared with [\[2\]](#), which is consistent with [\[9\]](#).

In examples, we write  $e(\mathbf{i}) = e(i_1 i_2 \dots i_n)$  if  $\mathbf{i} = (i_1, i_2, \dots, i_n)$ .

1.8. **Example.** Let  $n = 3$ ,  $e = 3$  and  $\Lambda = \Lambda_0$ . First,  $y_1 = 0$  because of the relations  $y_1^{(\Lambda_0, \alpha_{i_1})} e(\mathbf{i}) = 0$  and  $\sum_{\mathbf{i}} e(\mathbf{i}) = 0$ . It is not difficult to see that  $\psi_1 = 0 = y_2$  and that  $e(\mathbf{i}) = 0$  unless  $\mathbf{i} = (012)$  or  $(021)$ ; see, for example [\[18, Proposition 2.4.6\]](#). Hence,  $\mathcal{R}_e(\mathfrak{S}_3)$  is generated by  $\psi_2, y_3, e(012)$  and  $e(021)$ . Using the quadratic relation,

$$\psi_2^2 e(\mathbf{i}) = \begin{cases} -y_3 e(\mathbf{i}), & \text{if } \mathbf{i} = (012), \\ y_3 e(\mathbf{i}), & \text{if } \mathbf{i} = (021). \end{cases}$$

In turn, this implies that  $y_3^2 e(\mathbf{i}) = \pm y_3 \psi_2 e(\mathbf{i}) = \pm \psi_2 y_2 e(\mathbf{i}) = 0$ , so  $y_3^2 = 0$ . Therefore,  $\mathcal{R}_e(\mathfrak{S}_3)$  is spanned by

$$\{e(012), e(021), \psi_2 e(012), \psi_2 e(021), y_3 e(012), y_3 e(021)\}.$$

By [Theorem 1.9](#) below, these elements are a basis of  $\mathcal{R}_e(\mathfrak{S}_n)$ . ◇

To connect the algebras  $\mathcal{R}_e(\mathfrak{S}_n)$  and  $\mathcal{H}_\xi(\mathfrak{S}_n)$  define the **quantum characteristic** of  $\xi$  to be the smallest non-negative integer  $e$  such that  $1 + \xi + \dots + \xi^{e-1} = 0$ , and set  $e = \infty$  if no such integer exists. By definition,  $e \in \{2, 3, 4, 5, 6, \dots\} \cup \{\infty\}$  and  $e = 2$  if and only if  $\xi = -1$ .

Brundan and Kleshchev proved the following remarkable theorem, which connects the two algebras  $\mathcal{R}_e(\mathfrak{S}_n)$  and  $\mathcal{H}_\xi(\mathfrak{S}_n)$ .

1.9. **Theorem** (Brundan and Kleshchev [\[2\]](#), Rouquier [\[22, Corollary 3.20\]](#)).

*Suppose that  $F$  is a field and that  $\xi \neq -1$  has quantum characteristic  $e > 2$ . Then  $\mathcal{R}_e^F(\mathfrak{S}_n) \cong \mathcal{H}_\xi^F(\mathfrak{S}_n)$ .*

Hence, if  $F$  is a field then we can consider  $\mathcal{H}_\xi^F(\mathfrak{S}_n)$  as a  $\mathbb{Z}$ -graded algebra via the isomorphism  $\mathcal{H}_\xi^F(\mathfrak{S}_n) \cong \mathcal{R}_e^F(\mathfrak{S}_n)$ . We have stated a special case of Brundan and Kleshchev's result because this is all that we need. Over a field, Brundan and Kleshchev prove more generally that the cyclotomic Hecke algebras of type  $A$  are isomorphic to cyclotomic quiver Hecke algebras — and they also allow  $e = 2$ . To prove [Theorem 1.9](#) Brundan and Kleshchev construct an explicit isomorphism (in fact, they construct different isomorphisms for the cases when  $\xi = 1$  and  $\xi \neq 1$ ). Our proof of [Theorem B](#) builds from a variation on their ideas, following [\[9\]](#).

1.4. **Alternating quiver Hecke algebras of type  $A$ .** In this section we introduce a homogeneous analogue of the  $\#$ -involution of  $\mathcal{H}_\xi(\mathfrak{S}_n)$  and use it to define the alternating quiver Hecke algebras of type  $A$ .

If  $\mathbf{i} = (i_1, \dots, i_n) \in I^n$  let  $-\mathbf{i} = (-i_1, \dots, -i_n) \in I^n$ . Following [\[15, \(3.14\)\]](#), define  $\text{sgn}$  to be the unique automorphism of  $\mathcal{R}_e(\mathfrak{S}_n)$  such that

$$\psi_r^{\text{sgn}} = -\psi_r, \quad y_s^{\text{sgn}} = -y_s, \quad \text{and} \quad e(\mathbf{i})^{\text{sgn}} = e(-\mathbf{i}),$$

for  $1 \leq r < n$ ,  $1 \leq s \leq n$  and  $\mathbf{i} \in I^n$ .

If  $\alpha \in Q_n^+$  let  $\alpha'$  be the unique element of  $Q_n^+$  such that  $(\Lambda_i, \alpha) = (\Lambda_{-i}, \alpha')$ , for all  $i \in I$ . Recall from the introduction that  $\sim$  is the equivalence relation on  $I^n$  generated by  $\mathbf{i} \sim \mathbf{j}$  if  $\mathbf{j} = -\mathbf{i}$  and that if  $\alpha, \beta \in Q^+$  then  $\alpha \sim \beta$  if there exists  $\mathbf{i} \in I^\alpha$  such that  $-\mathbf{i} \in I^\beta$ . Hence,  $\alpha \sim \beta$  if and only if  $\beta \in \{\alpha, \alpha'\}$ .

Checking the relations in [Definition 1.5](#) reveals the following.



**1.10. Proposition** ([15, (3.14)]). *The map  $\mathbf{sgn}$  restricts to a homogeneous isomorphism of  $\mathbb{Z}$ -graded algebras  $\mathcal{R}_e(\mathfrak{S}_n)_\alpha \xrightarrow{\sim} \mathcal{R}_e(\mathfrak{S}_n)_{\alpha'}$ , for  $\alpha \in Q_n^+$ . Hence,  $\mathbf{sgn}$  is a homogeneous automorphism of  $\mathcal{R}_e(\mathfrak{S}_n)$  of order 2.*

Mirroring [Definition 1.3](#), we define the second algebra appearing in [Theorem A](#).

**1.11. Definition.** The **alternating quiver Hecke algebra** of  $\mathfrak{A}_n$  is the fixed-point subalgebra  $\mathcal{R}_e(\mathfrak{A}_n) = \mathcal{R}_e^{\mathbb{Z}}(\mathfrak{A}_n) = \{a \in \mathcal{R}_e(\mathfrak{S}_n) \mid a^{\mathbf{sgn}} = a\}$  of  $\mathcal{R}_e(\mathfrak{S}_n)$  under the involution  $\mathbf{sgn}$ .

Since  $\mathbf{sgn}$  is a homogeneous involution of  $\mathcal{R}_e(\mathfrak{S}_n)$ , an immediate and important consequence of [Definition 1.11](#) is the following.

**1.12. Corollary.** *The alternating quiver Hecke algebra  $\mathcal{R}_e(\mathfrak{A}_n)$  is a  $\mathbb{Z}$ -graded subalgebra of  $\mathcal{R}_e(\mathfrak{S}_n)$ .*

We finish this section with an example of how our main result, [Theorem A](#), works when  $n = 3 = e$ .

**1.13. Example.** Suppose that  $e = 3$  and  $n = 3$ . Then  $\mathfrak{A}_3 \cong \mathbb{Z}/3\mathbb{Z}$  is the cyclic group of order 3. By [Example 1.8](#),  $\mathcal{R}_3(\mathfrak{A}_3)$  is spanned by the three elements

$$1 = e(012) + e(021), \quad \Psi = \psi_2(e(012) - e(021)), \quad Y = y_3(e(012) - e(021)).$$

These three elements are homogeneous, with  $\deg \Psi = 1$  and  $\deg Y = 2$ , and [Theorem 1.9](#) implies that they are non-zero. Therefore,  $\{1, \Psi, Y\}$  is a basis of  $\mathcal{R}_3(\mathfrak{A}_3)$ . Using the relations in [Definition 1.5](#),  $\Psi^2 = -Y$  and  $\Psi^3 = -Y\Psi = 0$ . Therefore, the map  $\Psi \mapsto x$  determines an isomorphism of graded algebras

$$\mathcal{R}_e(\mathfrak{A}_3) = \langle \Psi \mid \Psi^3 = 0 \rangle \cong \mathbb{Z}[x]/(x^3),$$

where we put  $\deg x = 1$ . It is now easy to see that  $\mathcal{R}_3(\mathfrak{A}_3) \otimes \mathbb{F}_3 \cong \mathbb{F}_3\mathfrak{A}_3$ , where  $\mathbb{F}_3 = \mathbb{Z}/3\mathbb{Z}$ . For example, an isomorphism is determined by  $\Psi \mapsto 1 - s_1s_2$ .

The isomorphism above is not unique. In the special case when  $n = 3$  and  $\xi = 1 \in \mathbb{F}_3$ , the proof of [Theorem A](#) in [Section 2.5](#) constructs a different isomorphism  $\mathbb{F}_3\mathfrak{A}_3 \cong \mathcal{H}_\xi(\mathfrak{A}_3) \xrightarrow{\sim} \mathcal{R}_e(\mathfrak{A}_3)$  that is determined by  $\Psi \mapsto s_2s_1 - s_1s_2$ .  $\diamond$

The algebra  $\mathcal{R}_e(\mathfrak{S}_n)$  is defined in terms of the subalgebras  $\mathcal{R}_e(\mathfrak{S}_n)_\alpha$ . To give a presentation for  $\mathcal{R}_e(\mathfrak{A}_n)$  we need to work with the blocks of these algebras. As in the introduction, set  $Q_n^\varepsilon = Q_n^+/\sim$ . Using the notation introduced before [Proposition 1.10](#), if  $\alpha \in Q_n^+$  then  $\{\alpha, \alpha'\}$  is its  $\sim$ -equivalence class. If  $\gamma \in Q_n^\varepsilon$  set  $\mathcal{R}_e(\mathfrak{S}_n)_\gamma = \bigoplus_{\alpha \in \gamma} \mathcal{R}_e(\mathfrak{S}_n)_\alpha$ . By [Proposition 1.10](#),  $\mathbf{sgn}$  induces a homogeneous automorphism of  $\mathcal{R}_e(\mathfrak{S}_n)_\gamma$ . Define

$$(1.14) \quad \mathcal{R}_e(\mathfrak{A}_n)_\gamma = (\mathcal{R}_e(\mathfrak{S}_n)_\gamma)^{\mathbf{sgn}} = \{a \in \mathcal{R}_e(\mathfrak{S}_n)_\gamma \mid a = a^{\mathbf{sgn}}\}$$

to be the fixed-point subalgebra of  $\mathcal{R}_e(\mathfrak{S}_n)_\gamma$  under the  $\mathbf{sgn}$  automorphism. Since  $\mathcal{R}_e(\mathfrak{S}_n) = \bigoplus_\alpha \mathcal{R}_e(\mathfrak{S}_n)_\alpha$  we have the following decomposition of  $\mathcal{R}_e(\mathfrak{A}_n)$  as a direct sum of two-sided  $\mathbb{Z}$ -graded ideals.

**1.15. Corollary.** *As a graded algebra,  $\mathcal{R}_e(\mathfrak{A}_n) = \bigoplus_{\gamma \in Q_n^\varepsilon} \mathcal{R}_e(\mathfrak{A}_n)_\gamma$ .*



**1.5. A presentation for alternating quiver Hecke algebras of  $\mathfrak{A}_n$ .** In this section we prove [Theorem B](#). To do this we first give a “super” presentation for the quiver Hecke algebra  $\mathcal{R}_e(\mathfrak{S}_n)$ . For convenience, we identify the group  $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$  with  $\{0, 1\}$  in the obvious way.

We start by defining a new algebra  $\mathcal{R}_n^\varepsilon$  that, it turns out, is isomorphic to  $\mathcal{R}_e(\mathfrak{S}_n)$ . The advantage of  $\mathcal{R}_n^\varepsilon$  is that it is  $(\mathbb{Z}_2 \times \mathbb{Z})$ -graded where the  $\mathbb{Z}_2$ -grading encodes the effects of **sgn**. Abusing notation, we use similar notation for the generators of  $\mathcal{R}_e(\mathfrak{S}_n)$  and  $\mathcal{R}_n^\varepsilon$ . This is justified by [Proposition 1.19](#) below.

The sequence  $\mathbf{i} = (0, \dots, 0) \in I^n$ , which is the unique sequence such that  $\mathbf{i} = -\mathbf{i}$ , is potentially problematic for us. The next result resolves this.

**1.16. Lemma.** *Suppose that  $\mathbf{i} \in I^n$  and  $e(\mathbf{i}) \neq 0$ . Then  $i_1 = 0$  and  $i_2 = \pm 1$ . In particular,  $e(0, \dots, 0) = 0$ .*

*Proof.* By [\[7, Lemma 4.1c\]](#),  $e(\mathbf{i}) \neq 0$  if and only if  $\mathbf{i}$  is the residue sequence of some standard tableau (see [Section 2.1](#)), which readily implies the result. As we need this argument later, we give a direct proof following [\[18, Proposition 2.4.6\]](#). First,  $y_1 = 0$  and  $e(\mathbf{i}) \neq 0$  only if  $i_1 = 0$  by the cyclotomic relation  $y_1^{(\Lambda_0, \alpha_{i_1})} e(\mathbf{i}) = 0$ . If  $i_1 = i_2 = 0$  then  $e(\mathbf{i}) = (y_2 \psi_1 - \psi_1 y_1) e(\mathbf{i}) = y_2 \psi_1 e(\mathbf{i}) = y_2 e(\mathbf{i}) \psi_1$ , so that  $e(\mathbf{i}) = y_2^2 e(\mathbf{i}) \psi_1^2 = 0$ . Hence,  $e(0, 0, i_3, \dots, i_n) = 0$ . Finally, suppose that  $i_2 \neq \pm 1, 0$ . Then  $e(\mathbf{i}) = \psi_1^2 e(\mathbf{i}) = \psi_1 e(i_2, i_1, i_3, \dots, i_n) \psi_1 = 0$  since  $e(\mathbf{j}) = 0$  whenever  $j_1 \neq 0$ .  $\square$

Set  $I_+^n = \{\mathbf{i} \in I^n \mid i_1 = 0 \text{ and } i_2 = +1\}$  and  $I_-^n = \{\mathbf{i} \in I^n \mid i_1 = 0 \text{ and } i_2 = -1\}$ . Then [Lemma 1.16](#) shows that  $e(\mathbf{i}) \neq 0$  only if  $\mathbf{i} \in I_+^n \cup I_-^n$ .

Recall that if  $\gamma \in Q_n^\varepsilon$  then  $I^\gamma = \bigcup_{\alpha \in \gamma} I^\alpha$ .

**1.17. Definition.** Suppose that  $e \neq 2$ , 2 is invertible in  $\mathcal{Z}$  and that  $\gamma \in Q_n^\varepsilon$ . The algebra  $\mathcal{R}_\gamma^\varepsilon = \mathcal{R}_\gamma^\varepsilon(\Gamma_e, \Lambda_0)$  is the unital associative  $\mathcal{Z}$ -algebra with generators

$$\{\psi_r, y_s, \varepsilon_a(\mathbf{i}) \mid 1 \leq r < n, 1 \leq s \leq n, \mathbf{i} \in I^\gamma \text{ and } a \in \mathbb{Z}_2\}$$

subject to the relations

$$\begin{aligned} y_1^{(\Lambda_0, \alpha_{i_1})} \varepsilon_0(\mathbf{i}) &= 0, & \sum_{\mathbf{i} \in I^\gamma} \frac{1}{2} \varepsilon_0(\mathbf{i}) &= 1, & \varepsilon_0(\mathbf{i}) \varepsilon_0(\mathbf{j}) &= \delta_{\mathbf{i} \sim \mathbf{j}} \varepsilon_0(\mathbf{i}), \\ \varepsilon_a(\mathbf{i}) \varepsilon_b(\mathbf{i}) &= \varepsilon_{a+b}(\mathbf{i}), & \varepsilon_a(\mathbf{i}) &= (-1)^a \varepsilon_a(-\mathbf{i}), & \psi_r \varepsilon_a(\mathbf{i}) &= \varepsilon_a(s_r \cdot \mathbf{i}) \psi_r, \\ y_r \varepsilon_a(\mathbf{i}) &= \varepsilon_a(\mathbf{i}) y_r, & y_r y_s \varepsilon_1(\mathbf{i}) &= y_s y_r \varepsilon_1(\mathbf{i}), \\ \psi_r y_{r+1} \varepsilon_1(\mathbf{i}) &= (y_r \psi_r + \delta_{i_r, i_{r+1}}) \varepsilon_1(\mathbf{i}), & y_{r+1} \psi_r \varepsilon_1(\mathbf{i}) &= (\psi_r y_r + \delta_{i_r, i_{r+1}}) \varepsilon_1(\mathbf{i}), \\ \psi_r y_s \varepsilon_1(\mathbf{i}) &= y_s \psi_r \varepsilon_1(\mathbf{i}), & & \text{if } s \neq r, r+1, \\ \psi_r \psi_s \varepsilon_1(\mathbf{i}) &= \psi_s \psi_r \varepsilon_1(\mathbf{i}), & & \text{if } |r-s| > 1, \\ \psi_r^2 \varepsilon_1(\mathbf{i}) &= \begin{cases} 0, & \text{if } i_r = i_{r+1}, \\ (y_r - y_{r+1}) \varepsilon_0(\mathbf{i}), & \text{if } i_r \rightarrow i_{r+1}, \\ (y_{r+1} - y_r) \varepsilon_0(\mathbf{i}), & \text{if } i_r \leftarrow i_{r+1}, \\ \varepsilon_1(\mathbf{i}), & \text{otherwise,} \end{cases} \end{aligned}$$

$$\psi_r \psi_{r+1} \psi_r \varepsilon_0(\mathbf{i}) = \begin{cases} \psi_{r+1} \psi_r \psi_{r+1} \varepsilon_0(\mathbf{i}) - \varepsilon_1(\mathbf{i}), & \text{if } i_r = i_{r+2} \rightarrow i_{r+1}, \\ \psi_{r+1} \psi_r \psi_{r+1} \varepsilon_0(\mathbf{i}) + \varepsilon_1(\mathbf{i}), & \text{if } i_r = i_{r+2} \leftarrow i_{r+1}, \\ \psi_{r+1} \psi_r \psi_{r+1} \varepsilon_0(\mathbf{i}), & \text{otherwise,} \end{cases}$$

for  $\mathbf{i}, \mathbf{j} \in I^\gamma$ ,  $a, b \in \mathbb{Z}_2$  and all admissible  $r$  and  $s$ . Let  $\mathcal{R}_n^\varepsilon = \bigoplus_{\gamma \in Q_n^\varepsilon} \mathcal{R}_\gamma^\varepsilon$ .

It is routine to check that the relations in [Definition 1.17](#) are homogeneous with respect to the degree function  $\text{Deg} : \mathcal{R}_n^\varepsilon \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}$  that is determined by

$$\text{Deg } \psi_r \varepsilon_0(\mathbf{i}) = (1, -c_{i_r, i_{r+1}}), \quad \text{Deg } y_s = (1, 2) \quad \text{and} \quad \text{Deg } \varepsilon_a(\mathbf{i}) = (a, 0),$$

for  $1 \leq r < n$ ,  $1 \leq s \leq n$  and  $\mathbf{i} \in I^n$ . Hence,  $\mathcal{R}_n^\varepsilon$  is a  $(\mathbb{Z}_2 \times \mathbb{Z})$ -graded algebra.

By [Definition 1.17](#), if  $\mathbf{i} \in I^\gamma$  then  $\varepsilon_a(\mathbf{i}) = \pm \varepsilon_a(-\mathbf{i})$  so  $\mathcal{R}_n^\varepsilon$  is generated by the elements  $\{\psi_1, \dots, \psi_{n-1}\} \cup \{y_1, \dots, y_n\} \cup \{\varepsilon_a(\mathbf{i}) \mid a \in \mathbb{Z}_2 \text{ and } \mathbf{i} \in I_+^\gamma\}$ , where  $I_+^\gamma = I^\gamma \cap I_+^n$ . Similarly, set  $I_-^\gamma = I^\gamma \cap I_-^n$ . We use  $I^\gamma$  in [Definition 1.17](#) because it compactly encodes a sign change in the relation  $\psi_r \varepsilon_a(\mathbf{i}) = \varepsilon_a(s_r \cdot \mathbf{i}) \psi$  when  $r = 2$ .

We need a partial analogue of [Lemma 1.16](#) for  $\mathcal{R}_\gamma^\varepsilon$ .

**1.18. Lemma.** *Suppose that  $\mathbf{i} \in I^\gamma$  and that  $\mathbf{i} = -\mathbf{i}$ . Then  $\varepsilon_a(\mathbf{i}) = 0$ , for  $a \in \mathbb{Z}_2$ .*

*Proof.* If  $\mathbf{i} = -\mathbf{i}$  then  $\varepsilon_1(\mathbf{i}) = -\varepsilon_1(\mathbf{i}) = 0$ . Hence,  $\varepsilon_0(\mathbf{i}) = \varepsilon_1(\mathbf{i})\varepsilon_1(\mathbf{i}) = 0$ .  $\square$

By forgetting the  $\mathbb{Z}_2$ -grading on  $\mathcal{R}_\gamma^\varepsilon$  we obtain a  $\mathbb{Z}$ -graded algebra. Given the similarity of the relations in [Definition 1.5](#) and [Definition 1.17](#) the next result should not surprise the reader.

**1.19. Proposition.** *Suppose that  $\gamma \in Q_n^\varepsilon$ ,  $n \geq 0$ ,  $e \neq 2$  and that 2 is invertible in  $\mathcal{Z}$ . Then, as  $\mathbb{Z}$ -graded algebras,  $\mathcal{R}_\gamma^\varepsilon \cong \mathcal{R}_e(\mathfrak{S}_n)_\gamma$ .*

*Proof.* Define a map  $\theta$  from the generators of  $\mathcal{R}_\gamma^\varepsilon$  to  $\mathcal{R}_e(\mathfrak{S}_n)_\gamma$  by

$$\theta(\psi_r) = \psi_r, \quad \theta(y_s) = y_s, \quad \text{and} \quad \theta(\varepsilon_a(\mathbf{i})) = e(\mathbf{i}) + (-1)^a e(-\mathbf{i}),$$

for  $1 \leq r < n$ ,  $1 \leq s \leq n$ ,  $a \in \mathbb{Z}_2$  and  $\mathbf{i} \in I^\gamma$ . The relations in [Definition 1.17](#) are very similar to those of [Definition 1.5](#), so it is straightforward to check that  $\theta$  extends to an algebra homomorphism  $\mathcal{R}_\gamma^\varepsilon \rightarrow \mathcal{R}_e(\mathfrak{S}_n)_\gamma$ . By definition, if  $\mathbf{i} \in I^\gamma$  then  $e(\mathbf{i}) = \frac{1}{2}\theta(\varepsilon_0(\mathbf{i}) + \varepsilon_1(\mathbf{i}))$ . Therefore, the image of  $\theta$  contains all of the generators of  $\mathcal{R}_e(\mathfrak{S}_n)_\gamma$ . Hence,  $\theta$  is surjective.

Rather than proving directly that  $\theta$  is an isomorphism we define an inverse map. Define  $\vartheta$  to be the map from the set of non-zero generators of  $\mathcal{R}_e(\mathfrak{S}_n)_\gamma$  into  $\mathcal{R}_\gamma^\varepsilon$  given by

$$\vartheta(\psi_r) = \psi_r, \quad \vartheta(y_s) = y_s, \quad \text{and} \quad \vartheta(e(\mathbf{i})) = \frac{1}{2}(\varepsilon_0(\mathbf{i}) + \varepsilon_1(\mathbf{i})),$$

for  $1 \leq r < n$ ,  $1 \leq s \leq n$  and  $\mathbf{i} \in I^\gamma$ . If  $\mathbf{i} \in I^\gamma$  and  $a \in \mathbb{Z}_2$  then

$$\vartheta(e(\mathbf{i}) + (-1)^a e(-\mathbf{i})) = \frac{1}{2}(\varepsilon_0(\mathbf{i}) + \varepsilon_1(\mathbf{i}) + (-1)^a \varepsilon_0(-\mathbf{i}) + (-1)^a \varepsilon_1(-\mathbf{i})) = \varepsilon_a(\mathbf{i}).$$

Hence, by [Lemma 1.18](#), the image of  $\vartheta$  contains all of the generators of  $\mathcal{R}_\gamma^\varepsilon$  so, if it is a homomorphism, it is surjective.

Now, since  $\varepsilon_a(\mathbf{i})\varepsilon_b(\mathbf{i}) = \varepsilon_{a+b}(\mathbf{i})$ , for all  $\mathbf{i} \in I^\gamma$  and  $a, b \in \mathbb{Z}_2$ , by multiplying the relations in [Definition 1.17](#) on the right by  $\varepsilon_1(\mathbf{i})$  the following additional relations hold in  $\mathcal{R}_\gamma^\varepsilon$ :

$$\begin{aligned} y_r y_s \varepsilon_0(\mathbf{i}) &= y_s y_r \varepsilon_0(\mathbf{i}), \\ \psi_r y_{r+1} \varepsilon_0(\mathbf{i}) &= (y_r \psi_r + \delta_{i_r, i_{r+1}}) \varepsilon_0(\mathbf{i}), & y_{r+1} \psi_r \varepsilon_0(\mathbf{i}) &= (\psi_r y_r + \delta_{i_r, i_{r+1}}) \varepsilon_0(\mathbf{i}), \\ \psi_r y_s \varepsilon_0(\mathbf{i}) &= y_s \psi_r \varepsilon_0(\mathbf{i}), & & \text{if } s \neq r, r+1, \\ \psi_r \psi_s \varepsilon_0(\mathbf{i}) &= \psi_s \psi_r \varepsilon_0(\mathbf{i}), & & \text{if } |r-s| > 1, \end{aligned}$$

$$\psi_r^2 \varepsilon_0(\mathbf{i}) = \begin{cases} 0, & \text{if } i_r = i_{r+1}, \\ (y_r - y_{r+1}) \varepsilon_1(\mathbf{i}), & \text{if } i_r \rightarrow i_{r+1}, \\ (y_{r+1} - y_r) \varepsilon_1(\mathbf{i}), & \text{if } i_r \leftarrow i_{r+1}, \\ \varepsilon_0(\mathbf{i}), & \text{otherwise,} \end{cases}$$

$$\psi_r \psi_{r+1} \psi_r \varepsilon_1(\mathbf{i}) = \begin{cases} \psi_{r+1} \psi_r \psi_{r+1} \varepsilon_1(\mathbf{i}) - \varepsilon_0(\mathbf{i}), & \text{if } i_r = i_{r+2} \rightarrow i_{r+1}, \\ \psi_{r+1} \psi_r \psi_{r+1} \varepsilon_1(\mathbf{i}) + \varepsilon_0(\mathbf{i}), & \text{if } i_r = i_{r+2} \leftarrow i_{r+1}, \\ \psi_{r+1} \psi_r \psi_{r+1} \varepsilon_1(\mathbf{i}), & \text{otherwise,} \end{cases}$$

for all admissible  $r$  and  $s$  and  $\mathbf{i} \in I^\gamma$ . As it was for  $\theta$ , it is now straightforward to verify that  $\vartheta$  respects all of the relations of  $\mathcal{R}_e(\mathfrak{S}_n)_\gamma$ . Consequently,  $\vartheta$  extends to an algebra homomorphism  $\mathcal{R}_e(\mathfrak{S}_n)_\gamma \rightarrow \mathcal{R}_\gamma^\varepsilon$ .

In view of [Lemma 1.16](#) and [Lemma 1.18](#), the automorphisms  $\theta \circ \vartheta$  and  $\vartheta \circ \theta$  act as the identity on the non-zero generators of  $\mathcal{R}_e(\mathfrak{S}_n)_\gamma$  and  $\mathcal{R}_\gamma^\varepsilon$ , respectively. Therefore,  $\theta$  and  $\vartheta$  are mutually inverse isomorphisms and  $\mathcal{R}_\gamma^\varepsilon \xrightarrow{\cong} \mathcal{R}_e(\mathfrak{S}_n)_\gamma$  as (ungraded) algebras.

It remains to observe that  $\theta$  and  $\vartheta$  respect the  $\mathbb{Z}$ -gradings on both algebras, but this is immediate from the definitions of  $\theta$  and  $\vartheta$ . Hence,  $\mathcal{R}_e(\mathfrak{S}_n)_\gamma \cong \mathcal{R}_\gamma^\varepsilon$  as  $\mathbb{Z}$ -graded algebras, completing the proof.  $\square$

Define  $h \in \mathcal{R}_n^\varepsilon$  to be **even** if  $\text{Deg } h = (0, d)$  and  $h$  is **odd** if  $\text{Deg } h = (1, d)$ , for some  $d \in \mathbb{Z}$ . Let  $\mathcal{R}_\gamma^{\varepsilon+}$  and  $\mathcal{R}_\gamma^{\varepsilon-}$  be the sets of even and odd elements in  $\mathcal{R}_\gamma^\varepsilon$ , respectively. Then  $\mathcal{R}_\gamma^{\varepsilon+}$  is a subalgebra of  $\mathcal{R}_\gamma^\varepsilon$  and

$$(1.20) \quad \mathcal{R}_\gamma^\varepsilon = \mathcal{R}_\gamma^{\varepsilon+} \oplus \mathcal{R}_\gamma^{\varepsilon-}$$

as  $\mathcal{Z}$ -modules. Moreover, as we next show,  $\mathcal{R}_\gamma^{\varepsilon+}$  is isomorphic to  $\mathcal{R}_e(\mathfrak{A}_n)$  under the isomorphism of [Proposition 1.19](#).

**1.21. Corollary.** *Suppose that  $n \geq 0$  and that 2 is invertible in  $\mathcal{Z}$ . Let  $\gamma \in Q_n^\varepsilon$ . Then  $\mathcal{R}_\gamma^{\varepsilon+} \cong \mathcal{R}_e(\mathfrak{A}_n)_\gamma$  as  $\mathbb{Z}$ -graded algebras.*

*Proof.* Under the isomorphism  $\mathcal{R}_\gamma^\varepsilon \cong \mathcal{R}_e(\mathfrak{S}_n)_\gamma$  of [Proposition 1.19](#), the images of the even generators of  $\mathcal{R}_\gamma^\varepsilon$  are **sgn**-invariant and **sgn** multiplies the images of the odd generators by  $-1$ . Hence,  $\theta$  restricts to an isomorphism  $\mathcal{R}_\gamma^{\varepsilon+} \xrightarrow{\cong} \mathcal{R}_e(\mathfrak{A}_n)_\gamma$ .  $\square$

We can now prove [Theorem B](#) from the introduction.

*Proof of Theorem B.* Let  $A_\gamma$  be the abstract algebra with the presentation given in [Theorem B](#). By [Corollary 1.21](#), to prove [Theorem B](#) it is enough to show that  $A_\gamma \cong \mathcal{R}_\gamma^{\varepsilon+}$ . Define a map  $\Theta: A_\gamma \rightarrow \mathcal{R}_\gamma^{\varepsilon+}$  by

$$\Psi_r(\mathbf{i}) \mapsto \psi_r \varepsilon_1(\mathbf{i}), \quad Y_s(\mathbf{i}) \mapsto y_s \varepsilon_1(\mathbf{i}) \quad \text{and} \quad \varepsilon(\mathbf{i}) \mapsto \varepsilon_0(\mathbf{i}),$$

for all  $\mathbf{i} \in I^\gamma$ ,  $1 \leq r \leq n$  and  $1 \leq s < n$ . Using [Definition 1.17](#), and the relations in the proof of [Proposition 1.19](#), it is straightforward to check that all of the relations in  $A_\gamma$  are satisfied in  $\mathcal{R}_\gamma^{\varepsilon+}$ , so  $\Theta$  extends to an algebra homomorphism from  $A_\gamma$  to  $\mathcal{R}_\gamma^{\varepsilon+}$ .

By [Definition 1.17](#), the algebra  $\mathcal{R}_\gamma^{\varepsilon+}$  is generated by arbitrary products of the generators of  $\mathcal{R}_n^\varepsilon$  such that the resulting element is even. However, the only even generators of  $\mathcal{R}_n^\varepsilon$  are the idempotents  $\varepsilon_0(\mathbf{i})$ , for  $\mathbf{i} \in I^n$ , so  $\mathcal{R}_\gamma^{\varepsilon+}$  is generated by these idempotents together with all words of even length in the odd generators of  $\mathcal{R}_n^\varepsilon$ .

As  $\psi_r \varepsilon_1(\mathbf{i}) = \varepsilon_1(s_r \cdot \mathbf{j}) \psi_r$  and  $y_s \varepsilon_1(\mathbf{i}) = \varepsilon_1(\mathbf{i}) y_s$ , for admissible  $r, s$  and  $\mathbf{i} \in I^n$ , it follows that  $\mathcal{R}_\gamma^{\varepsilon+}$  is generated by the images of  $A_\gamma$  under  $\Theta$ . Hence,  $\Theta$  is surjective.

The algebra  $\mathcal{R}_\gamma^{\varepsilon+}$  is defined by generators and relations, so  $\mathcal{R}_\gamma^{\varepsilon+}$  is the subalgebra of  $\mathcal{R}_n^\varepsilon$  generated by the words of even length in the generators of  $\mathcal{R}_n^\varepsilon$  modulo the even part of the relational ideal that defines  $\mathcal{R}_n^\varepsilon$ . The only even relations for  $\mathcal{R}_n^\varepsilon$  are the idempotent relations and the commutation relations (these are the relations appearing in the first three lines of the relations in [Definition 1.17](#)). Hence, up to multiplication by an idempotent  $\varepsilon_0(\mathbf{i})$ , all of the even relations are essentially trivial. Therefore, the even component of the relational ideal for  $\mathcal{R}_n^\varepsilon$  is generated by words of even length in the odd relations for  $\mathcal{R}_n^\varepsilon$ . In turn, all of the even products of the odd relations are products of the even relations given in the proof of [Proposition 1.19](#), together with the even idempotent and commutation relations. All of these relations are the images under  $\Theta$  of the relations of  $A_\gamma$ . Hence,  $\Theta$  is an isomorphism and the result follows.  $\square$

## 2. THE SEMINORMAL FORM

To prove [Theorem A](#) we will work mainly in the setting of the semisimple representation theory of  $\mathcal{H}_\xi(\mathfrak{S}_n)$ . The idea is to show that the fixed-point subalgebras of  $\mathcal{H}_\xi(\mathfrak{A}_n) = \mathcal{H}_\xi(\mathfrak{S}_n)^\#$  and  $\mathcal{R}_e(\mathfrak{S}_n) = \mathcal{R}_e(\mathfrak{S}_n)^{\text{sgn}}$  coincide under the Brundan-Kleshchev isomorphism of [Theorem 1.9](#). Unfortunately, as shown by [Example 2.20](#), this is not true. To get around this we use the machinery developed in [\[9\]](#) to construct a new isomorphism  $\mathcal{R}_e(\mathfrak{S}_n) \xrightarrow{\sim} \mathcal{H}_\xi(\mathfrak{S}_n)$  that does restrict to an isomorphism  $\mathcal{R}_e(\mathfrak{A}_n) \xrightarrow{\sim} \mathcal{H}_\xi(\mathfrak{A}_n)$ .

**2.1. Tableau combinatorics.** This section recalls the partition and tableau combinatorics that are needed in this paper.

A **partition**  $\lambda = (\lambda_1, \lambda_2, \dots)$  is a weakly decreasing sequence of non-negative integers. The integers  $\lambda_r$  are the **parts** of  $\lambda$ , for  $r \geq 1$ , and  $\lambda$  is a partition of  $n$  if  $|\lambda| = n$ , where  $|\lambda| = \lambda_1 + \lambda_2 + \dots$ . Let  $\mathcal{P}_n$  be the set of partitions of  $n$ .

The **Young diagram** of a partition  $\lambda$  is the set  $\{(r, c) \mid 1 \leq c \leq \lambda_r \text{ for } r \geq 1\}$ , which we represent as a collection of left-justified boxes in the plane, with  $\lambda_r$  boxes in row  $r$  and with rows ordered from top to bottom by increasing row index. We identify a partition with its diagram. The partition  $\lambda'$  with  $\lambda'_r = \#\{c \geq 1 \mid \lambda_r \geq c\}$  is the partition **conjugate** to  $\lambda$ .

Suppose that  $\lambda \in \mathcal{P}_n$ . A  $\lambda$ -**tableau** is a bijective filling of the boxes of  $\lambda$  with the numbers  $1, 2, \dots, n$ . If  $\mathbf{t}$  is a  $\lambda$ -tableau then it has **shape**  $\lambda$  and we write  $\text{Shape}(\mathbf{t}) = \lambda$ . For  $m \geq 1$  let  $\mathbf{t}_{\downarrow m}$  be the subtableau of  $\mathbf{t}$  that contains the numbers  $1, 2, \dots, m$ .

A tableau is **standard** if its entries increase from left to right along each row and from top to bottom down each column. Hence,  $\mathbf{t}$  is standard if and only if  $\mathbf{t}_{\downarrow m}$  is standard for  $1 \leq m \leq n$ . Let  $\text{Std}(\lambda)$  be the set of standard  $\lambda$ -tableaux and let

$$\text{Std}(\mathcal{P}_n) = \bigcup_{\lambda \in \mathcal{P}_n} \text{Std}(\lambda) \quad \text{and} \quad \text{Std}^2(\mathcal{P}_n) = \bigcup_{\lambda \in \mathcal{P}_n} \text{Std}(\lambda) \times \text{Std}(\lambda).$$

If  $\mathbf{t}$  is a standard  $\lambda$ -tableau then the **conjugate** tableau  $\mathbf{t}'$  is the standard  $\lambda'$ -tableau obtained by swapping the rows and columns of  $\mathbf{t}$ .

The **initial  $\lambda$ -tableau**  $\mathbf{t}^\lambda$  is the  $\lambda$ -tableau obtained by inserting the numbers  $1, 2, \dots, n$  in order along the rows of  $\lambda$ , from left to right and then top to bottom.

The **co-initial** tableau  $\mathbf{t}_\lambda$  is the conjugate of  $\mathbf{t}^{\lambda'}$ . Then  $\mathbf{t}_\lambda$  is the unique  $\lambda$ -tableau that has the numbers  $1, 2, \dots, n$  entered in order down the columns of  $\lambda$ , from left to right.

Recall that  $I = \mathbb{Z}/e\mathbb{Z}$ . Let  $\mathbf{t}$  be a standard tableau and suppose that  $m$  appears in row  $r$  and column  $c$  of  $\mathbf{t}$ , where  $1 \leq m \leq n$ . The **content** and  **$e$ -residue** of  $m$  in  $\mathbf{t}$  are given by

$$c_m(\mathbf{t}) = c - r \in \mathbb{Z} \quad \text{and} \quad \text{res}_m(\mathbf{t}) = c - r + e\mathbb{Z} \in I.$$

respectively. The  **$e$ -residue sequence** of the tableau  $\mathbf{t}$  is the  $n$ -tuple

$$\text{res}(\mathbf{t}) = (\text{res}_1(\mathbf{t}), \dots, \text{res}_n(\mathbf{t})) \in I^n.$$

Given a sequence  $\mathbf{i} \in I^n$  let  $\text{Std}(\mathbf{i}) = \{\mathbf{t} \in \text{Std}(\mathcal{P}_n) \mid \text{res}(\mathbf{t}) = \mathbf{i}\}$  be the set of standard tableaux with residue sequence  $\mathbf{i}$ .

**2.2. Seminormal forms.** To prove [Theorem 2.45](#) we make extensive use of the semisimple representation theory of  $\mathcal{H}_\xi(\mathfrak{S}_n)$  using seminormal forms. This section introduces Jucys-Murphy elements and seminormal forms and proves some basic facts relating seminormal forms and the  $\#$ -involution.

Throughout this section we fix a field  $\mathcal{K}$  and a non-zero scalar  $t \in \mathcal{K}$ . Let  $\mathcal{H}_t^\mathcal{K}(\mathfrak{S}_n)$  be the Iwahori-Hecke algebra of  $\mathfrak{S}_n$  over  $\mathcal{K}$  with parameter  $t$ .

If  $k \in \mathbb{Z}$  define the **quantum integer** to be the scalar

$$[k]_t = \begin{cases} (1 + t + \dots + t^{k-1}), & \text{if } k \geq 0, \\ -(t^{-1} + t^{-2} + \dots + t^k), & \text{if } k < 0. \end{cases}$$

When  $t$  is understood we write  $[k] = [k]_t$ .

We need a well-known result, which is easily proved by induction on  $n$ . To state this, define the **Poincaré polynomial** of  $\mathcal{H}_t^\mathcal{K}(\mathfrak{S}_n)$  to be  $P_{\mathcal{H}}(t) = [1][2] \dots [n] \in \mathcal{K}$ .

**2.1. Lemma** (See [\[17, Lemma 3.34\]](#)). *Suppose that  $P_{\mathcal{H}}(t) \neq 0$  and that  $\mathbf{s}, \mathbf{t} \in \text{Std}(\mathcal{P}_n)$ . Then  $\mathbf{s} = \mathbf{t}$  if and only if  $[c_r(\mathbf{s})] = [c_r(\mathbf{t})]$ , for  $1 \leq r \leq n$ .*

We assume for the rest of this section that  $P_{\mathcal{H}}(t) \neq 0$ . In fact, the results that follow imply that  $\mathcal{H}_t^\mathcal{K}(\mathfrak{S}_n)$  is semisimple if and only if  $P_{\mathcal{H}}(t) \neq 0$  and, in turn, this is equivalent to the condition in [Lemma 2.1](#).

If  $\mathbf{t}$  is a tableau and  $1 \leq r \leq n$  then the **axial distance** from  $r+1$  to  $r$  in  $\mathbf{t}$  is

$$(2.2) \quad \rho_r(\mathbf{t}) = c_r(\mathbf{t}) - c_{r+1}(\mathbf{t}) \in \mathbb{Z}.$$

By definition,  $-n < \rho_r(\mathbf{t}) < n$  so  $[\rho_r(\mathbf{t})] \neq 0$  if  $P_{\mathcal{H}}(t) \neq 0$ .

The next definition will provide us with the framework to prove [Theorem 2.45](#).

**2.3. Definition** (Hu-Mathas [\[9, Definition 3.5\]](#)). Suppose that  $P_{\mathcal{H}}(t) \neq 0$ . A **\*-seminormal coefficient system** is a set of scalars

$$\alpha = \{\alpha_r(\mathbf{t}) \in \mathcal{K} \mid 1 \leq r < n \text{ and } \mathbf{t} \in \text{Std}(\mathcal{P}_n)\}$$

such that if  $\mathbf{t} \in \text{Std}(\mathcal{P}_n)$  and  $1 \leq r < n$  then:

- a)  $\alpha_r(\mathbf{t}) = 0$  whenever  $s_r \mathbf{t}$  is not standard.
- b)  $\alpha_r(\mathbf{t})\alpha_k(s_r \mathbf{t}) = \alpha_k(\mathbf{t})\alpha_r(s_k \mathbf{t})$  whenever  $1 \leq k < n$  and  $|r - k| > 1$ .
- c)  $\alpha_r(s_{r+1} s_r \mathbf{t})\alpha_{r+1}(s_r \mathbf{t})\alpha_r(\mathbf{t}) = \alpha_{r+1}(s_r s_{r+1} \mathbf{t})\alpha_r(s_{r+1} \mathbf{t})\alpha_{r+1}(\mathbf{t})$  if  $r \neq n-1$ ,
- d) if  $\mathbf{v} = s_r \mathbf{t} \in \text{Std}(\mathcal{P}_n)$  then

$$\alpha_r(\mathbf{t})\alpha_r(\mathbf{v}) = \frac{[1 + \rho_r(\mathbf{t})][1 + \rho_r(\mathbf{v})]}{[\rho_r(\mathbf{t})][\rho_r(\mathbf{v})]}.$$

Many examples of seminormal coefficient systems are given in [9, §3]. For example,  $\{\alpha_r(\mathbf{t})\}$  is seminormal coefficient system, where  $\alpha_r(\mathbf{t}) = \frac{[1+\rho_r(\mathbf{t})]}{[\rho_r(\mathbf{t})]}$  whenever  $\mathbf{t}, s_r \mathbf{t} \in \text{Std}(\mathcal{P}_n)$ . In Section 2.4 we fix a particular choice of seminormal coefficient system but until then we will work with an arbitrary coefficient system.

For  $k = 1, 2, \dots, n$  the **Jucys-Murphy** element  $L_k \in \mathcal{H}_t^{\mathcal{O}}$  is defined by

$$L_k = \sum_{j=1}^{k-1} t^{j-k} T_{(k-j,k)}.$$

A basis  $\{f_{\mathbf{st}} \mid (\mathbf{s}, \mathbf{t}) \in \text{Std}^2(\mathcal{P}_n)\}$  of  $\mathcal{H}_t^{\mathcal{K}}(\mathfrak{S}_n)$  is a **seminormal basis** if

$$L_k f_{\mathbf{st}} = [c_k(\mathbf{s})] f_{\mathbf{st}} \quad \text{and} \quad f_{\mathbf{st}} L_k = [c_k(\mathbf{t})] f_{\mathbf{st}},$$

for all  $(\mathbf{s}, \mathbf{t}) \in \text{Std}^2(\mathcal{P}_n)$  and  $1 \leq k \leq n$ . The basis  $\{f_{\mathbf{st}}\}$  is a **\*-seminormal basis** if, in addition,  $f_{\mathbf{st}}^* = f_{\mathbf{ts}}$ , for all  $(\mathbf{s}, \mathbf{t}) \in \text{Std}^2(\mathcal{P}_n)$ , where  $*$  is the unique anti-isomorphism of  $\mathcal{H}_t^{\mathcal{K}}(\mathfrak{S}_n)$  that fixes  $T_1, \dots, T_{n-1}$ .

Recall that  $P_{\mathcal{H}}(t) = [1][2] \dots [n]$ .

**2.4. Theorem** (The seminormal form [9, Theorem 3.9]). *Suppose that  $P_{\mathcal{H}}(t) \neq 0$  and that  $\alpha$  is a seminormal coefficient system for  $\mathcal{H}_t^{\mathcal{K}}(\mathfrak{S}_n)$ . Then:*

- a) *The algebra  $\mathcal{H}_t^{\mathcal{K}}(\mathfrak{S}_n)$  has a unique \*-seminormal basis  $\{f_{\mathbf{st}} \mid (\mathbf{s}, \mathbf{t}) \in \text{Std}^2(\mathcal{P}_n)\}$  such that*

$$f_{\mathbf{st}}^* = f_{\mathbf{ts}}, \quad L_k f_{\mathbf{st}} = [c_k(\mathbf{s})] f_{\mathbf{st}} \quad \text{and} \quad T_r f_{\mathbf{st}} = \alpha_r(\mathbf{s}) f_{\mathbf{ut}} - \frac{1}{[\rho_r(\mathbf{s})]} f_{\mathbf{st}},$$

where  $\mathbf{u} = (r, r+1)\mathbf{s}$ . (Set  $f_{\mathbf{ut}} = 0$  if  $\mathbf{u}$  is not standard.)

- b) *For  $\mathbf{t} \in \text{Std}(\mathcal{P}_n)$  there exist non-zero scalars  $\gamma_{\mathbf{t}} \in \mathcal{K}$  such that  $f_{\mathbf{st}} f_{\mathbf{uv}} = \delta_{\mathbf{tu}} \gamma_{\mathbf{t}} f_{\mathbf{sv}}$  and  $\{\frac{1}{\gamma_{\mathbf{t}}} f_{\mathbf{tt}} \mid \mathbf{t} \in \text{Std}(\mathcal{P}_n)\}$  is a complete set of pairwise orthogonal primitive idempotents.*

- c) *The \*-seminormal basis  $\{f_{\mathbf{st}} \mid (\mathbf{s}, \mathbf{t}) \in \text{Std}^2(\mathcal{P}_n)\}$  is uniquely determined by the \*-seminormal coefficient system  $\alpha$  and the scalars  $\{\gamma_{\mathbf{t}\lambda} \mid \lambda \in \mathcal{P}_n\}$ .*

By Theorem 2.4(b), if  $\mathbf{t} \in \text{Std}(\mathcal{P}_n)$  then  $F_{\mathbf{t}} = \frac{1}{\gamma_{\mathbf{t}}} f_{\mathbf{tt}}$  is a primitive idempotent in  $\mathcal{H}_t^{\mathcal{K}}(\mathfrak{S}_n)$ . As is well-known (see, for example, [9, (3.2)]),

$$F_{\mathbf{t}} = \prod_{k=1}^n \prod_{\substack{\mathbf{s} \in \text{Std}(\mathcal{P}_n) \\ c_k(\mathbf{s}) \neq c_k(\mathbf{t})}} \frac{L_k - [c_k(\mathbf{s})]}{[c_k(\mathbf{t})] - [c_k(\mathbf{s})]}.$$

In particular, the idempotent  $F_{\mathbf{t}}$  is independent of the choice of seminormal basis.

Let  $\mathcal{L}$  be the commutative subalgebra generated by the Jucys-Murphy elements. Theorem 2.4(a) implies that, as an  $(\mathcal{L}, \mathcal{L})$ -bimodule,  $\mathcal{H}_t^{\mathcal{K}}(\mathfrak{S}_n)$  decomposes as

$$(2.5) \quad \mathcal{H}_t^{\mathcal{K}}(\mathfrak{S}_n) = \bigoplus_{(\mathbf{s}, \mathbf{t}) \in \text{Std}^2(\mathcal{P}_n)} H_{\mathbf{st}},$$

where  $H_{\mathbf{st}} = \mathcal{K} f_{\mathbf{st}}$ . Equivalently,

$$H_{\mathbf{st}} = \{h \in \mathcal{H}_t^{\mathcal{K}}(\mathfrak{S}_n) \mid L_k h = [c_k(\mathbf{s})] h \text{ and } h L_k = [c_k(\mathbf{t})] h \text{ for } 1 \leq k \leq n\},$$

for  $(\mathbf{s}, \mathbf{t}) \in \text{Std}^2(\mathcal{P}_n)$ .

Let  $\#$  be the hash involution from (1.2) on  $\mathcal{H}_t^{\mathcal{K}}(\mathfrak{S}_n)$ . Then  $T_r^{\#} = -T_r + t - 1$ , for  $1 \leq r < n$ .

**2.6. Lemma.** *Suppose that  $1 \leq k \leq n$  and  $\mathbf{s} \in \text{Std}(\mathcal{P}_n)$ . Then*

$$L_k^\# f_{\mathbf{s}\mathbf{s}} = [c_k(\mathbf{s}')] f_{\mathbf{s}\mathbf{s}}.$$

*Proof.* For  $1 \leq k \leq n$  set  $\hat{L}_k = t^{1-k} T_{k-1} T_{k-2} \cdots T_2 T_1^2 T_2 \cdots T_{k-2} T_{k-1}$ . It is well-known and easy to prove that  $\hat{L}_k = (t-1)L_k + 1$ ; see, for example, [17, Exercise 3.6]. By Theorem 2.4(a),  $L_k f_{\mathbf{s}\mathbf{s}} = [c_k(\mathbf{s})] f_{\mathbf{s}\mathbf{s}}$ , so  $\hat{L}_k f_{\mathbf{s}\mathbf{s}} = t^{c_k(\mathbf{s})} f_{\mathbf{s}\mathbf{s}}$ . Now  $(\hat{L}_k)^\# = \hat{L}_k^{-1}$  since  $T_r^\# = -tT_r^{-1}$  by (1.2), for  $1 \leq r < k \leq n$ . Therefore,

$$\hat{L}_k^\# f_{\mathbf{s}\mathbf{s}} = \hat{L}_k^{-1} f_{\mathbf{s}\mathbf{s}} = t^{-c_k(\mathbf{s})} f_{\mathbf{s}\mathbf{s}} = t^{c_k(\mathbf{s}')} f_{\mathbf{s}\mathbf{s}},$$

where the last equality follows because  $c_k(\mathbf{s}') = -c_k(\mathbf{s})$  for  $1 \leq k \leq n$ . Hence,  $L_k^\# f_{\mathbf{s}\mathbf{s}} = [c_k(\mathbf{s}')] f_{\mathbf{s}\mathbf{s}}$  as claimed.  $\square$

**2.7. Lemma.** *Suppose that  $\mathbf{s} \in \text{Std}(\mathcal{P}_n)$ . Then  $F_{\mathbf{s}}^\# = F_{\mathbf{s}'}$ .*

*Proof.* Since  $F_{\mathbf{s}} = \frac{1}{\gamma_{\mathbf{s}}} f_{\mathbf{s}\mathbf{s}}$ , applying Lemma 2.6 gives

$$L_k F_{\mathbf{s}}^\# = (L_k^\# F_{\mathbf{s}})^\# = ([c_k(\mathbf{s}')] F_{\mathbf{s}})^\# = [c_k(\mathbf{s}')] F_{\mathbf{s}}^\#.$$

Similarly  $F_{\mathbf{s}}^\# L_k = [c_k(\mathbf{s}')] F_{\mathbf{s}}^\#$ . Therefore,  $F_{\mathbf{s}}^\# \in H_{\mathbf{s}'\mathbf{s}'}$  in the decomposition of (2.5). As  $F_{\mathbf{s}}$  is an idempotent, and  $\#$  is an algebra automorphism, it follows that  $F_{\mathbf{s}}^\# = F_{\mathbf{s}'}$  since this is the unique idempotent in  $H_{\mathbf{s}'\mathbf{s}'} = \mathcal{K} F_{\mathbf{s}'} = \mathcal{K} f_{\mathbf{s}'\mathbf{s}'}$ .  $\square$

**2.8. Corollary.** *Suppose that  $\mathbf{s} \in \text{Std}(\mathcal{P}_n)$ . Then  $f_{\mathbf{s}\mathbf{s}}^\# = \frac{\gamma_{\mathbf{s}}}{\gamma_{\mathbf{s}'}} f_{\mathbf{s}'\mathbf{s}'}$ .*

*Proof.* Using Theorem 2.4(b) and Lemma 2.7,  $f_{\mathbf{s}\mathbf{s}}^\# = \frac{1}{\gamma_{\mathbf{s}}} F_{\mathbf{s}}^\# = \frac{1}{\gamma_{\mathbf{s}}} F_{\mathbf{s}'} = \frac{\gamma_{\mathbf{s}'}}{\gamma_{\mathbf{s}}} f_{\mathbf{s}'\mathbf{s}'}$ .  $\square$

**2.9. Lemma.** *Let  $\mathbf{s}, \mathbf{u} \in \text{Std}(\mathcal{P}_n)$  be standard tableaux such that  $\mathbf{u} = (r, r+1)\mathbf{s}$ , for some integer  $r$  with  $1 \leq r < n$ . Then*

$$f_{\mathbf{u}\mathbf{s}}^\# = -\frac{\alpha_r(\mathbf{s}')\gamma_{\mathbf{s}}}{\alpha_r(\mathbf{s})\gamma_{\mathbf{s}'}} f_{\mathbf{u}'\mathbf{s}'}.$$

*Proof.* By Theorem 2.4(a),  $f_{\mathbf{u}\mathbf{s}} = \frac{1}{\alpha_r(\mathbf{s})} \left( T_r + \frac{1}{[\rho_r(\mathbf{s})]} \right) f_{\mathbf{s}\mathbf{s}}$ . Recall that  $T_r^\# = -tT_r^{-1} = -T_r + t - 1$ . Therefore, using (1.2) and Corollary 2.8 for the second equality,

$$\begin{aligned} f_{\mathbf{u}\mathbf{s}}^\# &= \frac{1}{\alpha_r(\mathbf{s})} \left( T_r + \frac{1}{[\rho_r(\mathbf{s})]} \right)^\# f_{\mathbf{s}\mathbf{s}}^\# \\ &= \frac{\gamma_{\mathbf{s}}}{\alpha_r(\mathbf{s})\gamma_{\mathbf{s}'}} \left( -T_r + t - 1 + \frac{1}{[\rho_r(\mathbf{s})]} \right) f_{\mathbf{s}'\mathbf{s}'} \\ &= -\frac{\gamma_{\mathbf{s}}}{\alpha_r(\mathbf{s})\gamma_{\mathbf{s}'}} \left( T_r - \frac{t^{\rho_r(\mathbf{s})}}{[\rho_r(\mathbf{s})]} \right) f_{\mathbf{s}'\mathbf{s}'} \\ &= -\frac{\gamma_{\mathbf{s}}}{\alpha_r(\mathbf{s})\gamma_{\mathbf{s}'}} \left( T_r + \frac{1}{[\rho_r(\mathbf{s}')] } \right) f_{\mathbf{s}'\mathbf{s}'}, \end{aligned}$$

since  $[\rho_r(\mathbf{s})] = -t^{\rho_r(\mathbf{s})} [-\rho_r(\mathbf{s})] = -t^{\rho_r(\mathbf{s})} [\rho_r(\mathbf{s}')]$ . Hence, the result follows by another application of Theorem 2.4(a).  $\square$

By Theorem 2.4(c) any  $*$ -seminormal basis is uniquely determined by a seminormal coefficient system and a choice of scalars  $\{\gamma_{\mathbf{t}\lambda} \mid \lambda \in \mathcal{P}_n\}$ . For completeness we determine these scalars for the seminormal basis  $\{f_{\mathbf{st}}^\# \mid (\mathbf{s}, \mathbf{t}) \in \text{Std}^2(\mathcal{P}_n)\}$ . Recall from Section 2.1 that  $\mathbf{t}_\lambda = (\mathbf{t}^{\lambda'})'$  is the co-initial  $\lambda$ -tableau.



**2.10. Proposition.** *The seminormal basis  $\{f_{st}^\# \mid (s, t) \in \text{Std}(\mathcal{P}_n)\}$  of  $\mathcal{H}_t^\mathcal{K}(\mathfrak{S}_n)$  is the seminormal basis determined by the seminormal coefficient system*

$$\{-\alpha_r(s') \mid s \in \text{Std}(\mathcal{P}_n) \text{ and } 1 \leq r < n\}$$

*together with the  $\gamma$ -coefficients  $\{\gamma_{t\lambda} \mid \lambda \in \mathcal{P}_n\}$ . That is, if  $(s, t) \in \text{Std}^2(\mathcal{P}_n)$  then*

$$L_k f_{st}^\# = [c_r(s')] f_{st}^\#, \quad f_{st}^\# L_k = [c_r(t')] f_{st}^\# \quad \text{and} \quad T_r f_{st}^\# = -\alpha_r(s) f_{ut}^\# - \frac{1}{[\rho_r(s')]} f_{st}^\#,$$

*where  $u = (r, r+1)s$ ,  $1 \leq k \leq n$  and  $1 \leq r < n$ . Moreover,  $f_{st}^\# f_{uv}^\# = \delta_{tu} \gamma_t f_{sv}^\#$ , for  $(s, t), (u, v) \in \text{Std}^2(\mathcal{P}_n)$ .*

*Proof.* If  $1 \leq k \leq n$  then  $L_k f_{st}^\# = [c_r(s')] f_{st}^\#$  and  $f_{st}^\# L_k = [c_r(t')] f_{st}^\#$  by Lemma 2.6. Using Theorem 2.4(a),

$$\begin{aligned} T_r f_{st}^\# &= (T_r^\# f_{st})^\# = ((-T_r + t - 1) f_{st})^\# \\ &= \left( -\alpha_r(s) f_{ut} + (t - 1 + \frac{1}{[\rho_r(s)]}) f_{st} \right)^\# \\ &= \left( -\alpha_r(s) f_{ut} - \frac{1}{[\rho_r(s')]} f_{st} \right)^\# \\ &= -\alpha_r(s) f_{ut}^\# - \frac{1}{[\rho_r(s')]} f_{st}^\#. \end{aligned}$$

Similarly,  $f_{st}^\# f_{uv}^\# = (f_{st} f_{uv})^\# = \delta_{tu} \gamma_t f_{sv}^\#$ . By Lemma 2.9,  $f_{st}^\# \in H_{s't'}$ , so the  $\alpha$ -coefficient corresponding to  $f_{st}^\#$  is naturally indexed by  $s'$  (and not by  $s$ ). Similarly, the labelling for the  $\gamma$ -coefficients involves conjugation because  $F_t = \frac{1}{\gamma_t} f_{t't'}^\#$  by Corollary 2.8. Hence, the result follows by Theorem 2.4.  $\square$

**2.3. Idempotent subrings and KLR generators.** We are almost ready to introduce the generators of  $\mathcal{H}_t^\mathcal{O}$  that we need to prove Theorem 2.45. This section defines roughly half of these generators. The definition of these elements involves lifting idempotents from the non-semisimple case to the semisimple case and to do this we need to place additional constraints upon the rings that we work over.

If  $\mathcal{O}$  is a ring let  $\mathcal{J} = \mathcal{J}(\mathcal{O})$  be the Jacobson radical of  $\mathcal{O}$ .

**2.11. Definition** ([9, Definition 4.1]). Suppose that  $\mathcal{O}$  is a subring of a field  $\mathcal{K}$  and that  $t \in \mathcal{O}^\times$ . The pair  $(\mathcal{O}, t)$  is an  *$e$ -idempotent subring* of  $\mathcal{K}$  if:

- a) The Poincaré polynomial  $P_{\mathcal{H}}(t)$  is a non-zero element of  $\mathcal{O}$ ,
- b) If  $k \notin e\mathbb{Z}$  then  $[k]_t$  is invertible in  $\mathcal{O}$ ,
- c) If  $k \in e\mathbb{Z}$  then  $[k]_t \in \mathcal{J}(\mathcal{O})$ .

Condition (a) ensures that Lemma 2.1 and Theorem 2.4 apply and, in particular, that  $\mathcal{H}_t^\mathcal{K}(\mathfrak{S}_n)$  has a seminormal basis  $\{f_{st}\}$ . As discussed in [7, Example 4.2], when considering the Hecke algebra  $\mathcal{H}_\xi(\mathfrak{S}_n)$  defined over the field  $F$  with parameter  $\xi \in F^\times$ , one natural choice of idempotent subring is to let  $x$  be an indeterminate over  $F$  and set  $\mathcal{K} = F(x)$ ,  $t = x + \xi$  and  $\mathcal{O} = F[x, x^{-1}]_{(x)}$ . Note that  $\mathfrak{m} = x\mathcal{O}$  is the unique maximal ideal of  $\mathcal{O}$  and that  $(\mathcal{K}, \mathcal{O}, F)$  is a modular system with  $\mathcal{H}_\xi(\mathfrak{S}_n) \cong \mathcal{H}_t^\mathcal{O} \otimes_{\mathcal{O}} F$ , where  $F$  is considered as an  $\mathcal{O}$ -module by letting  $x$  act as multiplication by 0.

The hash involution  $\#$  from (1.2) is well-defined on  $\mathcal{H}_t^\mathcal{O}$ . Let  $\mathcal{H}_t^\mathcal{O}(\mathfrak{A}_n)$  the  $\mathcal{O}$ -subalgebra of  $\#$ -fixed points in  $\mathcal{H}_t^\mathcal{O}$ . Then  $\mathcal{H}_\xi(\mathfrak{A}_n) \cong \mathcal{H}_t^\mathcal{O}(\mathfrak{A}_n) \otimes_{\mathcal{O}} F$ .

Recall that if  $\mathbf{i} \in I^n$  then  $\text{Std}(\mathbf{i}) = \{\mathbf{t} \in \text{Std}(\mathcal{P}_n) \mid \text{res}(\mathbf{t}) = \mathbf{i}\}$ . Using an idea that goes back to Murphy [21], the **i-residue idempotent** is defined to be the element

$$f_{\mathbf{i}}^{\mathcal{O}} = \sum_{\mathbf{t} \in \text{Std}(\mathbf{i})} \frac{1}{\gamma_{\mathbf{t}}} f_{\mathbf{t}\mathbf{t}} = \sum_{\mathbf{t} \in \text{Std}(\mathbf{i})} F_{\mathbf{t}}.$$

By Theorem 2.4(b), if  $\mathbf{s} \in \text{Std}(\mathbf{j})$  and  $\mathbf{t} \in \text{Std}(\mathbf{k})$  are tableaux of the same shape then  $f_{\mathbf{i}}^{\mathcal{O}} f_{\mathbf{st}} = \delta_{\mathbf{ij}} f_{\mathbf{st}}$  and  $f_{\mathbf{st}} f_{\mathbf{i}}^{\mathcal{O}} = \delta_{\mathbf{ik}} f_{\mathbf{st}}$ , for  $\mathbf{i}, \mathbf{j}, \mathbf{k} \in I^n$ .

By definition,  $f_{\mathbf{i}}^{\mathcal{O}} \in \mathcal{H}_t^{\mathcal{K}}(\mathfrak{S}_n)$  but, in fact,  $f_{\mathbf{i}}^{\mathcal{O}} \in \mathcal{H}_t^{\mathcal{O}}$ .

**2.12. Lemma.** *Suppose that  $\mathbf{i} \in I^n$ . Then  $f_{\mathbf{i}}^{\mathcal{O}} \in \mathcal{H}_t^{\mathcal{O}}$  and  $(f_{\mathbf{i}}^{\mathcal{O}})^{\#} = f_{-\mathbf{i}}^{\mathcal{O}}$ .*

*Proof.* Since  $(\mathcal{O}, t)$  is an idempotent subring,  $f_{\mathbf{i}}^{\mathcal{O}} \in \mathcal{H}_t^{\mathcal{O}}$  by [9, Lemma 4.5]. To prove that  $(f_{\mathbf{i}}^{\mathcal{O}})^{\#} = f_{-\mathbf{i}}^{\mathcal{O}}$  first observe that  $\mathbf{s} \in \text{Std}(\mathbf{i})$  if and only if  $\mathbf{s}' \in \text{Std}(-\mathbf{i})$ . Therefore, by Lemma 2.7,

$$(f_{\mathbf{i}}^{\mathcal{O}})^{\#} = \sum_{\mathbf{s} \in \text{Std}(\mathbf{i})} F_{\mathbf{s}}^{\#} = \sum_{\mathbf{s} \in \text{Std}(\mathbf{i})} F_{\mathbf{s}'} = f_{-\mathbf{i}}^{\mathcal{O}}$$

as claimed.  $\square$

Following [2, 9], define  $M_r = 1 - L_r + tL_{r+1}$ , for  $1 \leq r \leq n$ . If  $(\mathbf{s}, \mathbf{t}) \in \text{Std}^2(\mathcal{P}_n)$  then it follows easily using Theorem 2.4(a) that

$$(2.13) \quad M_r f_{\mathbf{st}} = t^{c_r(\mathbf{s})} [1 - \rho_r(\mathbf{s})] f_{\mathbf{st}}.$$

The next result says that these elements are invertible when projected onto certain residue idempotents  $f_{\mathbf{i}}^{\mathcal{O}}$ , for  $\mathbf{i} \in I^n$ .

**2.14. Corollary** ([9, Corollary 4.8]). *Suppose that  $\mathbf{i} \in I^n$  and  $i_r \neq i_{r+1} + 1$ , for  $1 \leq r < n$  and . Then*

$$\sum_{\mathbf{s} \in \text{Std}(\mathbf{i})} \frac{t^{-c_r(\mathbf{t})}}{[1 - \rho_r(\mathbf{s})]} F_{\mathbf{s}} \in \mathcal{H}_t^{\mathcal{O}}.$$

In view of Corollary 2.14, if  $i_r \neq i_{r+1} + 1$  define the formal symbol

$$\frac{1}{M_r} f_{\mathbf{i}}^{\mathcal{O}} = \sum_{\mathbf{s} \in \text{Std}(\mathbf{i})} \frac{t^{-c_r(\mathbf{t})}}{[1 - \rho_r(\mathbf{s})]} F_{\mathbf{s}}.$$

This abuse of notation is justified because  $\frac{1}{M_r} f_{\mathbf{i}}^{\mathcal{O}} M_r = f_{\mathbf{i}}^{\mathcal{O}}$  by (2.13). We will use these elements to define the KLR-generators of  $\mathcal{H}_t^{\mathcal{O}}$  that we use to prove Theorem A.

The results of [9] depend upon choosing an arbitrary section of the natural quotient map  $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/e\mathbb{Z}$ . In this paper we are far less flexible and need to use a particular section of this map. If  $i \in I$  let  $\hat{i} \geq 0$  be the smallest non-negative integer such that  $i = \hat{i} + e\mathbb{Z}$ . (If  $e = \infty$  set  $\hat{i} = i$ .) This defines an embedding  $I \hookrightarrow \mathbb{Z}; i \mapsto \hat{i}$ . For  $\mathbf{i} \in I^n$  set  $\rho_r(\mathbf{i}) = \hat{i}_r - \hat{i}_{r+1}$ , for  $1 \leq r < n$ .

By Theorem 2.4, the identity element of  $\mathcal{H}_t^{\mathcal{O}}$  can be written as  $1 = \sum_{\mathbf{i} \in I^n} f_{\mathbf{i}}^{\mathcal{O}}$ . So if  $h \in \mathcal{H}_t^{\mathcal{O}}$  then  $h = \sum_{\mathbf{i} \in I^n} h f_{\mathbf{i}}^{\mathcal{O}}$  is uniquely determined by its projection onto the idempotents  $f_{\mathbf{i}}^{\mathcal{O}}$ .

**2.15. Definition** ([9, Definition 4.14]). Fix an integer  $1 \leq r < n$  and define the element  $\psi_r^+ = \sum_{\mathbf{i} \in I^n} \psi_r^+ f_{\mathbf{i}}^{\mathcal{O}}$  by

$$\psi_r^+ f_{\mathbf{i}}^{\mathcal{O}} = \begin{cases} (1 + T_r) \frac{t^{i_r}}{M_r} f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_r = i_{r+1}, \\ (T_r L_r - L_r T_r) t^{-i_r} f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_r \rightarrow i_{r+1}, \\ (T_r L_r - L_r T_r) \frac{1}{M_r} f_{\mathbf{i}}^{\mathcal{O}}, & \text{otherwise.} \end{cases}$$

For  $1 \leq s \leq n$  define  $y_s^+ = \sum_{\mathbf{i} \in I^n} t^{-i_s} (L_s - [i_s]) f_{\mathbf{i}}^{\mathcal{O}}$ .

Recall from [Section 1.5](#) that  $Q_n^\varepsilon = Q_n^+ / \sim$ . For  $\alpha \in Q^+$  define

$$\mathcal{H}_\alpha^{\mathcal{O}} = \mathcal{H}_t^{\mathcal{O}} f_\alpha^{\mathcal{O}}, \quad \text{where} \quad f_\alpha^{\mathcal{O}} = \sum_{\mathbf{i} \in I^\alpha} f_{\mathbf{i}}^{\mathcal{O}}.$$

For  $\gamma \in Q_n^\varepsilon$  set  $f_\gamma^{\mathcal{O}} = \sum_{\alpha \in \gamma} f_\alpha^{\mathcal{O}}$  and set  $\mathcal{H}_\gamma^{\mathcal{O}} = \bigoplus_{\alpha \in \gamma} \mathcal{H}_\alpha^{\mathcal{O}} = \mathcal{H}_t^{\mathcal{O}} f_\gamma^{\mathcal{O}}$ .

**2.16. Proposition.** Suppose that  $(\mathcal{O}, t)$  is an idempotent subring. Then  $f_\gamma^{\mathcal{O}}$  is a central idempotent in  $\mathcal{H}_t^{\mathcal{O}}$  and  $\mathcal{H}_t^{\mathcal{O}} = \bigoplus_{\gamma \in Q_n^\varepsilon} \mathcal{H}_\gamma^{\mathcal{O}}$ .

*Proof.* By [Lemma 2.12](#),  $f_\gamma^{\mathcal{O}} \in \mathcal{H}_t^{\mathcal{O}}$  and it follows from [Theorem 2.4](#) that  $f_\gamma^{\mathcal{O}}$  is a central idempotent and that  $1 = \sum_{\gamma \in Q_n^\varepsilon} f_\gamma^{\mathcal{O}}$ . Hence,  $\mathcal{H}_\gamma^{\mathcal{O}} = f_\gamma^{\mathcal{O}} \mathcal{H}_t^{\mathcal{O}} f_\gamma^{\mathcal{O}}$  is a subalgebra of  $\mathcal{H}_t^{\mathcal{O}}$  and  $\mathcal{H}_t^{\mathcal{O}} = \bigoplus_{\gamma \in Q_n^\varepsilon} \mathcal{H}_\gamma^{\mathcal{O}}$ .  $\square$

By [12], for  $\alpha \in Q_n^+$  the algebras  $\mathcal{H}_\alpha^{\mathcal{O}} \otimes_{\mathcal{O}} F$  are indecomposable two-sided ideals of  $\mathcal{H}_\xi(\mathfrak{S}_n)$ . Later we need the counterpart of [Corollary 1.15](#) for  $\mathcal{H}_\xi(\mathfrak{A}_n)$ . If  $\gamma \in Q_n^\varepsilon$  then  $(f_\gamma^{\mathcal{O}})^\# = f_\gamma^{\mathcal{O}}$  by [Lemma 2.12](#). Therefore,  $\#$  restricts to an automorphism of  $\mathcal{H}_\gamma^{\mathcal{O}}$ . Define

$$(2.17) \quad \mathcal{H}_t^{\mathcal{O}}(\mathfrak{A}_n)_\gamma = (\mathcal{H}_\gamma^{\mathcal{O}})^\# = \{h \in \mathcal{H}_\gamma^{\mathcal{O}} \mid h^\# = h\} = \mathcal{H}_t^{\mathcal{O}}(\mathfrak{A}_n) f_\gamma^{\mathcal{O}}.$$

As  $1 = \sum_\gamma f_\gamma^{\mathcal{O}}$ , [Proposition 2.16](#) immediately implies the following.

**2.18. Corollary.** Suppose that  $(\mathcal{O}, t)$  is an idempotent subring. Then

$$\mathcal{H}_t^{\mathcal{O}}(\mathfrak{A}_n) = \bigoplus_{\gamma \in Q_n^\varepsilon} \mathcal{H}_t^{\mathcal{O}}(\mathfrak{A}_n)_\gamma.$$

The subalgebra  $\mathcal{H}_t^{\mathcal{O}}(\mathfrak{A}_n)_\gamma$  is a block of  $\mathcal{H}_t^{\mathcal{O}}(\mathfrak{A}_n)$  in the sense that it is a two-sided ideal and a direct summand of  $\mathcal{H}_t^{\mathcal{O}}(\mathfrak{A}_n)$ . Let  $F$  be a field that is an  $\mathcal{O}$ -algebra and set  $\mathcal{H}_\xi^F(\mathfrak{A}_n)_\gamma = \mathcal{H}_t^{\mathcal{O}}(\mathfrak{A}_n)_\gamma \otimes_{\mathcal{O}} F$ . Then  $\mathcal{H}_\xi^F(\mathfrak{A}_n)_\gamma$  is almost always indecomposable. See [Theorem 3.22](#) for the precise statement.

**2.19. Theorem** (Hu-Mathas [9, Theorem A]). Suppose that  $\mathcal{K}$  is a field,  $\gamma \in Q_n^+$  and that  $(\mathcal{O}, t)$  an  $e$ -idempotent subring of  $\mathcal{K}$ , where  $e \neq 2$ . As an  $\mathcal{O}$ -algebra, the Iwahori-Hecke algebra  $\mathcal{H}_\gamma^{\mathcal{O}}$  is generated by the elements

$$\{f_{\mathbf{i}}^{\mathcal{O}} \mid \mathbf{i} \in I^\gamma\} \cup \{\psi_r^+ \mid 1 \leq r < n\} \cup \{y_s^+ \mid 1 \leq s \leq n\}$$

subject to the relations

$$\begin{aligned} (y_1^+)^{(\Lambda_0, \alpha_{i_1})} f_{\mathbf{i}}^{\mathcal{O}} &= 0, & f_{\mathbf{i}}^{\mathcal{O}} f_{\mathbf{j}}^{\mathcal{O}} &= \delta_{\mathbf{ij}} f_{\mathbf{i}}^{\mathcal{O}}, & \sum_{\mathbf{i} \in I^\gamma} f_{\mathbf{i}}^{\mathcal{O}} &= 1 \\ y_r^+ f_{\mathbf{i}}^{\mathcal{O}} &= f_{\mathbf{i}}^{\mathcal{O}} y_r^+, & \psi_r^+ f_{\mathbf{i}}^{\mathcal{O}} &= f_{s_r \cdot \mathbf{i}}^{\mathcal{O}} \psi_r^+, & y_r^+ y_s^+ &= y_s^+ y_r^+ \\ \psi_r^+ y_{r+1}^+ f_{\mathbf{i}}^{\mathcal{O}} &= (y_r^+ \psi_r^+ + \delta_{i_r, i_{r+1}}) f_{\mathbf{i}}^{\mathcal{O}}, & y_{r+1}^+ \psi_r^+ f_{\mathbf{i}}^{\mathcal{O}} &= (\psi_r^+ y_r^+ + \delta_{i_r, i_{r+1}}) f_{\mathbf{i}}^{\mathcal{O}} \end{aligned}$$

$$\begin{aligned}
 \psi_r^+ y_s^+ &= y_s^+ \psi_r^+, & \text{if } s \neq r, r+1, \\
 \psi_r^+ \psi_s^+ &= \psi_s^+ \psi_r^+, & \text{if } |r-s| > 1, \\
 (\psi_r^+)^2 f_{\mathbf{i}}^{\mathcal{O}} &= \begin{cases} (y_r^{\langle 1+\rho_r(\mathbf{i}) \rangle} - y_{r+1}^+) f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_r \rightarrow i_{r+1}, \\ (y_{r+1}^{\langle 1-\rho_r(\mathbf{i}) \rangle} - y_r^+) f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_r \leftarrow i_{r+1}, \\ 0, & \text{if } i_r = i_{r+1}, \\ f_{\mathbf{i}}^{\mathcal{O}}, & \text{otherwise,} \end{cases} \\
 \psi_r^+ \psi_{r+1}^+ \psi_r^+ f_{\mathbf{i}}^{\mathcal{O}} &= \begin{cases} (\psi_{r+1}^+ \psi_r^+ \psi_{r+1}^+ - t^{1+\rho_r(\mathbf{i})}) f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_r = i_{r+2} \rightarrow i_{r+1}, \\ (\psi_{r+1}^+ \psi_r^+ \psi_{r+1}^+ + 1) f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_r = i_{r+2} \leftarrow i_{r+1}, \\ \psi_{r+1}^+ \psi_r^+ \psi_{r+1}^+ f_{\mathbf{i}}^{\mathcal{O}}, & \text{otherwise,} \end{cases}
 \end{aligned}$$

where  $y_r^{\langle d \rangle} f_{\mathbf{i}}^{\mathcal{O}} = (t^d y_r^+ - [d]) f_{\mathbf{i}}^{\mathcal{O}}$  for  $d \in \mathbb{Z}$ , and  $\rho_r(\mathbf{i}) = \hat{i}_r - \hat{i}_{r+1}$ , for  $\mathbf{i}, \mathbf{j} \in I^\gamma$  and all admissible  $r, s$ .

*Proof.* By [9, Theorem A], this result holds for the algebras  $\mathcal{H}_\alpha^{\mathcal{O}}$ , for  $\alpha \in Q_n^+$ . As  $\mathcal{H}_\gamma^{\mathcal{O}} = \bigoplus_{\alpha \in \gamma} \mathcal{H}_\alpha^{\mathcal{O}}$  this gives the result for  $\mathcal{H}_\gamma^{\mathcal{O}}$ . The proof of [9, Theorem A] assumes that  $e < \infty$  (or  $e = 0$  in the notation of [9]), however, this assumption is only needed for [9, (3.2)] which is automatic in level 1 when  $\Lambda = \Lambda_0$ . Alternatively, as in [9, Corollary 2.15], it is enough to consider the case when  $n < e < \infty$ .  $\square$

As above, let  $\mathfrak{m}$  be a maximal ideal of  $\mathcal{O}$  and set  $\mathbb{F} = \mathcal{O}/\mathfrak{m}$  and  $\xi = t + \mathfrak{m} \in \mathbb{F}$  and let  $\mathcal{H}_\xi(\mathfrak{S}_n)$  be the Iwahori-Hecke algebra over  $\mathbb{F}$  with parameter  $\xi$ . Then  $\mathcal{H}_\xi^{\mathbb{F}}(\mathfrak{S}_n) \cong \mathcal{H}_t^{\mathcal{O}} \otimes_{\mathcal{O}} \mathbb{F}$ . The definition of an  $e$ -idempotent subring ensures that  $\xi$  has quantum characteristic  $e$ . Comparing the relations in Definition 1.5 with those in Theorem 2.19, modulo  $\mathfrak{m}$ , there is an algebra isomorphism  $\theta: \mathcal{R}_e^{\mathbb{F}}(\mathfrak{S}_n) \rightarrow \mathcal{H}_\xi^{\mathbb{F}}(\mathfrak{S}_n)$  determined by

$$\psi_r \mapsto \psi_r^+ \otimes 1_{\mathbb{F}}, \quad y_r \mapsto y_r^+ \otimes 1_{\mathbb{F}} \quad \text{and} \quad e(\mathbf{i}) \mapsto f_{\mathbf{i}}^{\mathcal{O}} \otimes 1_{\mathbb{F}},$$

for all admissible  $r$  and  $\mathbf{i} \in I^n$ . Unfortunately, as the next example shows, we cannot use  $\theta$  to prove Theorem A because  $\theta \circ \text{sgn} \neq \# \circ \theta$ .

**2.20. Example.** Suppose that  $n = 3$ ,  $\Lambda = \Lambda_0$  and work over  $\mathbb{F}_3$ , the field with three elements. By Example 1.8,

$$\{e(012), e(021), y_3 e(012), y_3 e(021), \psi_2 e(012), \psi_2 e(021)\}$$

is a basis of  $\mathcal{R}_e(\mathfrak{S}_3) \cong \mathbb{F}_3 \mathfrak{S}_3$  and, by Example 1.13,

$$\{e(012) + e(021), \psi_2(e(012) - e(021)), y_3(e(012) - e(021))\}$$

is a basis of  $\mathcal{R}_e(\mathfrak{A}_3)$ . Let  $\theta: \mathcal{R}_e(\mathfrak{S}_3) \rightarrow \mathbb{F}_3 \mathfrak{S}_3$  be the Brundan-Kleshchev isomorphism induced by Theorem 2.19. With some work it is possible to show that:

$$\begin{aligned}
 \theta(e(012) + e(021)) &= 1 \\
 \theta(y_3(e(012) - e(021))) &= 1 + s_1 s_2 + s_2 s_1 \\
 \theta(\psi_2(e(012) - e(021))) &= s_2 + 2s_1 s_2 s_1.
 \end{aligned}$$

In particular,  $\theta$  does not restrict to an isomorphism between  $\mathcal{R}_e(\mathfrak{A}_3)$  and  $\mathbb{F}_3 \mathfrak{A}_3$ .  $\diamond$

To obtain an isomorphism  $\mathcal{R}_e(\mathfrak{S}_n) \rightarrow \mathcal{H}_\xi(\mathfrak{S}_n)$ , which restricts to an isomorphism  $\mathcal{R}_e(\mathfrak{A}_n) \rightarrow \mathcal{H}_\xi(\mathfrak{A}_n)$ , we modify the generators of  $\mathcal{H}_t^{\mathcal{O}}$  given in Theorem 2.19.

**2.21. Definition.** Let  $\psi_r^- = (\psi_r^+)^{\#}$  and  $y_s^- = (y_s^+)^{\#}$ , for  $1 \leq r < n$  and  $1 \leq s \leq n$ .

Notice that  $(f_i^\mathcal{O})^\# = f_{-i}^\mathcal{O}$  by Lemma 2.12. Therefore, since  $\#$  is an automorphism, the elements  $\{\psi_r^-, y_s^-, f_i^\mathcal{O}\}$  generate  $\mathcal{H}_t^\mathcal{O}$ , subject to essentially the same relations as those given in Theorem 2.19 except that  $\mathbf{i}$  should be replaced with  $-\mathbf{i}$ . As we need this result below we state it in full for easy reference.

**2.22. Corollary.** *Suppose that  $e > 2$ ,  $\gamma \in Q_n^+$  and  $n \geq 0$ . Let  $(\mathcal{O}, t)$  be an  $e$ -idempotent subring of  $\mathcal{K}$ . Then  $\mathcal{H}_\gamma^\mathcal{O}$  is generated as an  $\mathcal{O}$ -algebra by the elements*

$$\{\psi_r^- \mid 1 \leq r < n\} \cup \{y_s^- \mid 1 \leq s \leq n\} \cup \{f_i^\mathcal{O} \mid \mathbf{i} \in I^\gamma\}$$

subject to the relations

$$\begin{aligned} (y_1^-)^{(\Lambda_0, \alpha_{i_1})} f_i^\mathcal{O} &= 0, & f_i^\mathcal{O} f_j^\mathcal{O} &= \delta_{ij} f_i^\mathcal{O}, & \sum_{\mathbf{i} \in I^\gamma} f_i^\mathcal{O} &= 1 \\ y_r^- f_i^\mathcal{O} &= f_i^\mathcal{O} y_r^-, & \psi_r^- f_i^\mathcal{O} &= f_{s_r \cdot \mathbf{i}}^\mathcal{O} \psi_r^-, & y_r^- y_s^- &= y_s^- y_r^- \\ \psi_r^- y_{r+1}^- f_i^\mathcal{O} &= (y_r^- \psi_r^- + \delta_{ir, i_{r+1}}) f_i^\mathcal{O}, & y_{r+1}^- \psi_r^- f_i^\mathcal{O} &= (\psi_r^- y_r^- + \delta_{ir, i_{r+1}}) f_i^\mathcal{O} \\ \psi_r^- y_s^- &= y_s^- \psi_r^-, & & \text{if } s \neq r, r+1, \\ \psi_r^- \psi_s^- &= \psi_s^- \psi_r^-, & & \text{if } |r-s| > 1, \\ (\psi_r^-)^2 f_i^\mathcal{O} &= \begin{cases} (y_r^{\langle 1-\rho_r(\mathbf{i}) \rangle} - y_{r+1}^-) f_i^\mathcal{O}, & \text{if } i_r \leftarrow i_{r+1}, \\ (y_{r+1}^{\langle 1+\rho_r(\mathbf{i}) \rangle} - y_r^-) f_i^\mathcal{O}, & \text{if } i_r \rightarrow i_{r+1}, \\ 0, & \text{if } i_r = i_{r+1}, \\ f_i^\mathcal{O}, & \text{otherwise,} \end{cases} \\ \psi_r^- \psi_{r+1}^- \psi_r^- f_i^\mathcal{O} &= \begin{cases} (\psi_{r+1}^- \psi_r^- \psi_{r+1}^- - t^{1-\rho_r(\mathbf{i})}) f_i^\mathcal{O}, & \text{if } i_r = i_{r+2} \leftarrow i_{r+1}, \\ (\psi_{r+1}^- \psi_r^- \psi_{r+1}^- + 1) f_i^\mathcal{O}, & \text{if } i_r = i_{r+2} \rightarrow i_{r+1}, \\ \psi_{r+1}^- \psi_r^- \psi_{r+1}^- f_i^\mathcal{O}, & \text{otherwise,} \end{cases} \end{aligned}$$

where  $y_r^{\langle d \rangle} f_i^\mathcal{O} = (t^d y_r^- - [d]) f_i^\mathcal{O}$ , for all  $d \in \mathbb{Z}$ , for  $\mathbf{i}, \mathbf{j} \in I^\gamma$  and all admissible  $r$  and  $s$ .

The elements  $y_r^{\langle d \rangle}$  appearing in Theorem 2.19 and Corollary 2.22 are different. In the next section we introduce a third variation of this notation. The meaning of  $y_r^{\langle d \rangle}$  will always be clear from context.

**2.4. Signed KLR generators.** This section sets up the machinery that will be used to construct the isomorphism  $\mathcal{R}_e(\mathfrak{A}_n) \rightarrow \mathcal{H}_\xi(\mathfrak{A}_n)$ . The idea is to use the results of the last two sections to give a new presentation of  $\mathcal{H}_t^\mathcal{O}$ , which induces an isomorphism  $\mathcal{R}_e(\mathfrak{S}_n) \rightarrow \mathcal{H}_\xi(\mathfrak{S}_n)$  that restricts to an isomorphism  $\mathcal{R}_e(\mathfrak{A}_n) \rightarrow \mathcal{H}_\xi(\mathfrak{A}_n)$ . To do this we use the generators of  $\mathcal{H}_t^\mathcal{O}$  given in Theorem 2.19 and Corollary 2.22 together with a particular seminormal coefficient system and idempotent subring.

The seminormal coefficient system that we use to prove Theorem A forces us to work over a ring that contains “enough” square roots. We start by defining this ring, following [19, Definition 3.1]. Suppose that the Iwahori-Hecke algebra  $\mathcal{H}$  is defined over the field  $F$  with parameter  $\xi \in F$  of quantum characteristic  $e$ . Recall that in this paper we are assuming that characteristic of  $F$  is not 2 and that  $e > 2$ .

**2.23. Definition.** Let  $x$  be an indeterminate over  $F$  and set  $t = x + \xi$ . In the algebraic closure  $\overline{F(x)}$  of  $F(x)$  fix square roots  $\sqrt{-1}$ ,  $\sqrt{t}$  and  $\sqrt{[h]}$ , for  $1 < h \leq n$ . Let

$$\mathcal{O} = F[\sqrt{t}, \sqrt{[h]} \mid 1 < h \leq n]_{(x)}$$

be the localization of  $F[\sqrt{t}, \sqrt{[h]} \mid 1 < h \leq n]$  at the maximal ideal generated by  $x$ . Let  $\mathcal{K}$  be the field of fractions of  $\mathcal{O}$ .

Note that  $t$  is invertible in  $\mathcal{O}$  so that we can consider the Iwahori-Hecke algebra  $\mathcal{H}_t^\mathcal{O}$  with parameter  $t$ . By [19, Corollary 5.12], the field of fractions  $\mathcal{K}$  of  $\mathcal{O}$  is a splitting field for the semisimple algebra  $\mathcal{H}_t^\mathcal{K}(\mathfrak{A}_n)$ .

Let  $\mathfrak{m} = x\mathcal{O}$  be the maximal ideal of  $\mathcal{O}$  and set  $\mathbb{F} = \mathcal{O}/\mathfrak{m}$ . Then  $F$  is (isomorphic to) a subfield of  $\mathbb{F}$ . Moreover,  $\xi$  is identified with the image of  $t$  under the natural map  $\mathcal{O} \rightarrow \mathbb{F}$ . Hence,  $\mathcal{H}_\xi(\mathfrak{S}_n) \otimes_F \mathbb{F} \cong \mathcal{H}_t^\mathcal{O} \otimes_{\mathcal{O}} \mathbb{F}$ . (Note that working over  $\mathbb{F}$  does not change the representation theory of  $\mathcal{H}_\xi(\mathfrak{S}_n)$  because any field is a splitting field for  $\mathcal{H}_\xi(\mathfrak{S}_n)$  since it is a cellular algebra [6, 17].) By construction,  $\mathbb{F}$  contains square roots  $\sqrt{-1}$ ,  $\sqrt{\xi}$  and  $\sqrt{[h]_\xi}$ , for  $-n \leq h \leq n$ . In general,  $\mathbb{F}$  is a non-trivial extension of  $F$ .

For  $0 < h \leq n$  fix a choice of “negative” square roots in  $\mathcal{O}$  by setting

$$(2.24) \quad \sqrt{[-h]} = \sqrt{-1}(\sqrt{t})^{-h} \sqrt{[h]}.$$

Then  $\sqrt{[-h]} \in \mathcal{O}$  for  $-n \leq h \leq n$ . If  $h > 0$  then  $[-h] = -t^{-h}[h]$  so the effect of (2.24) is to fix the sign of the square root of  $[-h]$ .

In order to apply the results of Theorem 2.19 and Corollary 2.22 we need to check that  $(\mathcal{O}, t)$  is an idempotent subring in the sense of Definition 2.11. Part (a) of Definition 2.11 is automatic whereas parts (b) and (c) follow from the observation that if  $k \in \mathbb{Z}$  then the polynomial  $[k] = [k]_t \in \mathcal{O}$  has zero constant term, as a polynomial in  $x$ , if and only if  $k \in e\mathbb{Z}$ . Hence, we have the following.

**2.25. Lemma.** *The pair  $(\mathcal{O}, t)$  is an idempotent subring.*

Now that we have fixed an idempotent subring we turn to the proof of Theorem A. The idea is to use the generators of  $\mathcal{H}_\gamma^\mathcal{O}$  from Theorem 2.19 for “half” of  $\mathcal{H}_\gamma^\mathcal{O}$  and to use the generators from Corollary 2.22 the rest of the time. To make this more precise, recall from after Lemma 1.16 that

$$I_+^n = \{\mathbf{i} \in I_n \mid i_1 = 0 \text{ and } i_2 = 1\} \quad \text{and} \quad I_-^n = \{\mathbf{i} \in I_n \mid i_1 = 0 \text{ and } i_2 = -1\}.$$

These sets are disjoint because  $e \neq 2$ . Set  $I_+^\gamma = I^\gamma \cap I_+^n$  and  $I_-^\gamma = I^\gamma \cap I_-^n$ , for  $\gamma \in Q_n^\varepsilon$ . Then the map  $\mathbf{i} \mapsto -\mathbf{i}$  is a bijection of sets  $I_+^\gamma \xrightarrow{\sim} I_-^\gamma$ .

**2.26. Lemma.** *Suppose that  $\mathbf{i} \in I^\gamma$  and  $f_\mathbf{i}^\mathcal{O} \neq 0$ . Then  $\mathbf{i} \in I_+^\gamma$  or  $\mathbf{i} \in I_-^\gamma$ .*

*Proof.* By definition,  $f_\mathbf{i}^\mathcal{O} \neq 0$  only if  $\text{Std}(\mathbf{i}) \neq \emptyset$  or, equivalently,  $\mathbf{i} = \text{res}(\mathbf{t})$  for some standard tableau  $\mathbf{t} \in \text{Std}(\mathcal{P}_n)$ . If  $\mathbf{t} \in \text{Std}(\mathbf{i})$  then  $i_1 = \text{res}_1(\mathbf{t}) = 0$  and  $i_2 = \text{res}_2(\mathbf{t}) = \pm 1$ , so  $\mathbf{i} \in I_+^\gamma \cup I_-^\gamma$ .  $\square$

In particular, if  $h \in \mathcal{H}_\gamma^\mathcal{O}$  then  $h = \sum_{\mathbf{i} \in I_+^\gamma} (h f_\mathbf{i}^\mathcal{O} + h f_{-\mathbf{i}}^\mathcal{O})$ . In what follows we apply Lemma 2.26, and this observation, without further mention.

Motivated in part by Proposition 2.10 we make the following definition.

**2.27. Definition.** An **alternating coefficient system** is a  $*$ -seminormal coefficient system  $\alpha = \{\alpha_r(\mathbf{t})\}$  such that  $\alpha_r(\mathbf{t}) = -\alpha_r(\mathbf{t}')$ , for  $1 \leq r < n$  and  $\mathbf{t} \in \text{Std}(\mathcal{P}_n)$ .

Consider the case when  $n = 3$  and  $\alpha$  is an alternating coefficient system. Let  $\mathbf{t} = \begin{bmatrix} 1 & 2 \\ 3 \end{bmatrix}$  and  $\mathbf{s} = \begin{bmatrix} 1 & 3 \\ 2 \end{bmatrix}$ , so that  $\text{res}(\mathbf{t}) \in I_+^3$ . By Definition 2.27 and Definition 2.3,

$$\alpha_2(\mathbf{t})^2 = -\alpha_2(\mathbf{s})\alpha_2(\mathbf{t}) = -\frac{t[3]}{[2]^2}.$$

So  $\alpha_2(\mathbf{t}) = \pm\sqrt{-1}\sqrt{t}\sqrt{[3]}/[2]$ . In the argument that follows we need  $\alpha_2(\mathbf{t})$  to have such values for all  $\mathbf{t} \in \text{Std}(\mathcal{P}_n)$ . Following [19, §3], for  $\mathbf{i} \in I^n$  and  $1 \leq r < n$  define

$$(2.28) \quad \alpha_r(\mathbf{t}) = \begin{cases} \frac{t^{\rho_r(\mathbf{t})/2} \sqrt{[1+\rho_r(\mathbf{t})]} \sqrt{[1-\rho_r(\mathbf{t})]}}{[\rho_r(\mathbf{t})]}, & \text{if } \mathbf{i} \in I_+^n, \\ -\alpha_r(\mathbf{t}'), & \text{if } \mathbf{i} \in I_-^n, \\ 0, & \text{otherwise.} \end{cases}$$

By Definition 2.23,  $\alpha_r(\mathbf{t}) \in \mathcal{O}$  for all  $\mathbf{t} \in \text{Std}(\mathcal{P}_n)$  and  $1 \leq r < n$ . Moreover, if  $\mathbf{i} \in I_\pm^n$  and  $\mathbf{t} \in \text{Std}(\mathbf{i})$  then  $\alpha_2(\mathbf{t}) = \pm\sqrt{-1}\sqrt{t}\sqrt{[3]}/[2]$  by (2.24).

It is straightforward to check that  $\{\alpha_r(\mathbf{t})\}$  is an alternating coefficient system. In particular, if  $\mathbf{t}$  is standard,  $1 \leq r < n$  and  $s_r \mathbf{t}$  is not standard then  $\rho_r(\mathbf{t}) = \pm 1$  so that  $\alpha_r(\mathbf{t}) = 0$ .

Using Theorem 2.4, we fix an arbitrary seminormal basis  $\{f_{\mathbf{st}}\}$  for  $\mathcal{H}_t^{\mathcal{K}}(\mathfrak{S}_n)$  that is compatible with the seminormal coefficient system defined by (2.28). Note that Definition 2.29, and hence the results that follow, do not depend on this choice of seminormal basis.

**2.29. Definition.** Suppose that  $1 \leq r < n$  and  $1 \leq s \leq n$ . If  $r \neq 2$  define

$$\psi_r^{\mathcal{O}} = \sum_{\mathbf{i} \in I_+^\gamma} (\psi_r^+ f_{\mathbf{i}}^{\mathcal{O}} - \psi_r^- f_{-\mathbf{i}}^{\mathcal{O}}) \quad \text{and} \quad y_s^{\mathcal{O}} = \sum_{\mathbf{i} \in I_+^\gamma} (y_s^+ f_{\mathbf{i}}^{\mathcal{O}} - y_s^- f_{-\mathbf{i}}^{\mathcal{O}}),$$

and when  $r = 2$  set  $\psi_2^{\mathcal{O}} = \sum_{\mathbf{i} \in I_+^\gamma} \kappa_{\mathbf{i}} (\psi_2^+ f_{\mathbf{i}}^{\mathcal{O}} - \psi_2^- f_{-\mathbf{i}}^{\mathcal{O}})$ , where

$$\kappa_{\mathbf{i}} = \begin{cases} t^{-1}, & \text{if } e = 3, \\ \frac{\sqrt{t}}{\sqrt{[3]}}, & \text{if } e > 3. \end{cases}$$

For convenience, set  $\kappa_{-\mathbf{i}} = \kappa_{\mathbf{i}}$ , for  $\mathbf{i} \in I_+^\gamma$ . The scalars  $\kappa_{\mathbf{i}}$  are needed to ensure that  $\psi_2^{\mathcal{O}}$  satisfies analogues of the quadratic and braid relations in Definition 1.5.

By Definition 2.27,  $\kappa_{\mathbf{i}}$  is invertible in  $\mathcal{O}$  for all  $\mathbf{i} \in I^\gamma$ . The reason why  $\kappa_{\mathbf{i}}$  depends only on  $e$ , and not on the quiver  $\Gamma_e$ , goes back to Lemma 1.16: if  $\mathbf{i} \in I^\gamma$  and  $e(\mathbf{i}) \neq 0$  or, equivalently,  $f_{\mathbf{i}}^{\mathcal{O}} \neq 0$  then the possible values for  $i_1, i_2$  and  $i_3$  are tightly constrained.

By Definition 2.23, the elements in  $\{\psi_r^{\mathcal{O}} \mid 1 \leq r < n\} \cup \{y_s^{\mathcal{O}} \mid 1 \leq s \leq n\}$  belong to  $\mathcal{H}_\gamma^{\mathcal{O}}$ . The aim is now to show that these elements, together with the idempotents  $\{f_{\mathbf{i}}^{\mathcal{O}} \mid \mathbf{i} \in I^n\}$ , generate  $\mathcal{H}_\gamma^{\mathcal{O}}$  subject to relations that are similar to those in Theorem 2.19. This will imply that these elements induce an isomorphism  $\mathcal{R}_e(\mathfrak{S}_n) \otimes_{\mathbb{Z}} \mathbb{F} \cong \mathcal{H}_\xi(\mathfrak{S}_n) \otimes_F \mathbb{F}$ . Before we start the proof we note the following consequence of Definition 2.29 and Lemma 2.12. Ultimately, this observation will imply that  $\mathcal{H}_\xi(\mathfrak{A}_n) \otimes_F \mathbb{F} \cong \mathcal{R}_e(\mathfrak{A}_n) \otimes_{\mathbb{Z}} \mathbb{F}$ .

**2.30. Corollary.** Suppose that  $1 \leq r < n$ ,  $1 \leq s \leq n$  and  $\mathbf{i} \in I^\gamma$ . Then

$$(\psi_r^{\mathcal{O}})^\# = -\psi_r^{\mathcal{O}}, \quad (y_s^{\mathcal{O}})^\# = -y_s^{\mathcal{O}} \quad \text{and} \quad (f_{\mathbf{i}}^{\mathcal{O}})^\# = f_{-\mathbf{i}}^{\mathcal{O}}.$$

The first step is to give a new generating set for  $\mathcal{H}_\gamma^{\mathcal{O}}$ .

**2.31. Proposition.** Suppose  $\gamma \in Q_n^\varepsilon$ . Then  $\mathcal{H}_\gamma^{\mathcal{O}}$  is generated by

$$\{\psi_r^{\mathcal{O}} \mid 1 \leq r < n\} \cup \{y_s^{\mathcal{O}} \mid 1 \leq s \leq n\} \cup \{f_{\mathbf{i}}^{\mathcal{O}} \mid \mathbf{i} \in I^\gamma\}.$$



*Proof.* Let  $H_\gamma$  be the  $\mathcal{O}$ -subalgebra of  $\mathcal{H}_\gamma^\mathcal{O}$  generated by the elements in the statement of the proposition. It is enough to show that  $T_r f_{\mathbf{i}}^\mathcal{O} \in H_\gamma$ , for  $1 \leq r < n$  and  $\mathbf{i} \in I^\gamma$ , since these elements generate  $\mathcal{H}_\gamma^\mathcal{O}$ . Further, by [Corollary 2.30](#),  $H_\gamma^\# = H_\gamma$  so it is enough to show that  $T_r f_{\mathbf{i}}^\mathcal{O} \in H_\gamma$ , for  $\mathbf{i} \in I_+^\gamma$  and  $1 \leq r < n$ . Let  $f_+^\mathcal{O} = \sum_{\mathbf{i} \in I_+^\gamma} f_{\mathbf{i}}^\mathcal{O}$ . As remarked above,  $\kappa_{\mathbf{i}}$  is an invertible scalar in  $\mathcal{O}$ . Therefore, the  $\mathcal{O}$ -module  $H_\gamma f_+^\mathcal{O}$  contains the elements  $\{\psi_r^+ f_{\mathbf{i}}^\mathcal{O}, y_s^+ f_{\mathbf{i}}^\mathcal{O}, f_{\mathbf{i}}^\mathcal{O} \mid \mathbf{i} \in I_+^\gamma\}$ . Hence,  $H_\gamma f_+^\mathcal{O} = \mathcal{H}_t^\mathcal{O} f_+^\mathcal{O}$  by [Theorem 2.19](#). This completes the proof.  $\square$

For the rest of this paper, for  $d \in \mathbb{Z}$ ,  $1 \leq r \leq n$  and  $\mathbf{i} \in I^\gamma$  we set

$$(2.32) \quad y_r^{(d)} f_{\mathbf{i}}^\mathcal{O} = \begin{cases} (t^d y_r^\mathcal{O} - [d]) f_{\mathbf{i}}^\mathcal{O}, & \text{if } \mathbf{i} \in I_+^\gamma, \\ (t^d y_r^\mathcal{O} + [d]) f_{\mathbf{i}}^\mathcal{O}, & \text{if } \mathbf{i} \in I_-^\gamma. \end{cases}$$

Since  $y_r^\mathcal{O} = \sum_{\mathbf{i} \in I_+^\gamma} (y_r^+ f_{\mathbf{i}}^\mathcal{O} - y_r^- f_{\mathbf{i}}^\mathcal{O})$  this is compatible with the two definitions of  $y_r^{(d)} f_{\mathbf{i}}^\mathcal{O}$  used in the last section

The rest of this section determines a set of defining relations for  $\mathcal{H}_\gamma^\mathcal{O}$  for the generators of  $\mathcal{H}_\gamma^\mathcal{O}$  from [Proposition 2.31](#). Fortunately, much of the work has already been done because [Theorem 2.19](#) and [Corollary 2.22](#) give us a large number of relations. More precisely, they give the following list of relations, *not* involving  $\psi_2^\mathcal{O}$ .

**2.33. Lemma.** *Suppose that  $\gamma \in Q_n^\varepsilon$ . The following identities hold in  $\mathcal{H}_\gamma^\mathcal{O}$ :*

$$\begin{aligned} (y_1^\mathcal{O})^{(\Lambda_0, \alpha_{i_1})} f_{\mathbf{i}}^\mathcal{O} &= 0, & f_{\mathbf{i}}^\mathcal{O} f_{\mathbf{j}}^\mathcal{O} &= \delta_{\mathbf{ij}} f_{\mathbf{i}}^\mathcal{O}, & \sum_{\mathbf{i} \in I^\gamma} f_{\mathbf{i}}^\mathcal{O} &= 1 \\ y_t^\mathcal{O} f_{\mathbf{i}}^\mathcal{O} &= f_{\mathbf{i}}^\mathcal{O} y_t^\mathcal{O}, & \psi_r^\mathcal{O} f_{\mathbf{i}}^\mathcal{O} &= f_{s_r \cdot \mathbf{i}}^\mathcal{O} \psi_r^\mathcal{O}, & y_r^\mathcal{O} y_t^\mathcal{O} &= y_t^\mathcal{O} y_r^\mathcal{O} \\ \psi_r^\mathcal{O} y_{r+1}^\mathcal{O} f_{\mathbf{i}}^\mathcal{O} &= (y_r^\mathcal{O} \psi_r^\mathcal{O} + \delta_{i_r, i_{r+1}}) f_{\mathbf{i}}^\mathcal{O}, & y_{r+1}^\mathcal{O} \psi_r^\mathcal{O} f_{\mathbf{i}}^\mathcal{O} &= (\psi_r^\mathcal{O} y_r^\mathcal{O} + \delta_{i_r, i_{r+1}}) f_{\mathbf{i}}^\mathcal{O} \\ \psi_r^\mathcal{O} y_t^\mathcal{O} &= y_t^\mathcal{O} \psi_r^\mathcal{O}, & & \text{if } t \neq r, r+1, \\ \psi_r^\mathcal{O} \psi_s^\mathcal{O} &= \psi_s^\mathcal{O} \psi_r^\mathcal{O}, & & \text{if } |r-s| > 1, \\ (\psi_r^\mathcal{O})^2 f_{\mathbf{i}}^\mathcal{O} &= \begin{cases} (y_r^{(1+\rho_r(\mathbf{i}))} - y_{r+1}^\mathcal{O}) f_{\mathbf{i}}^\mathcal{O}, & \text{if } i_r \rightarrow i_{r+1} \text{ and } \mathbf{i} \in I_+^\gamma, \\ (y_{r+1}^\mathcal{O} - y_r^{(1-\rho_r(\mathbf{i}))}) f_{\mathbf{i}}^\mathcal{O}, & \text{if } i_r \leftarrow i_{r+1} \text{ and } \mathbf{i} \in I_-^\gamma, \\ (y_{r+1}^{(1-\rho_r(\mathbf{i}))} - y_r^\mathcal{O}) f_{\mathbf{i}}^\mathcal{O}, & \text{if } i_r \leftarrow i_{r+1} \text{ and } \mathbf{i} \in I_+^\gamma, \\ (y_r^\mathcal{O} - y_{r+1}^{(1+\rho_r(\mathbf{i}))}) f_{\mathbf{i}}^\mathcal{O}, & \text{if } i_r \rightarrow i_{r+1} \text{ and } \mathbf{i} \in I_-^\gamma, \\ 0, & \text{if } i_r = i_{r+1}, \\ f_{\mathbf{i}}^\mathcal{O}, & \text{otherwise,} \end{cases} \\ \psi_r^\mathcal{O} \psi_{r+1}^\mathcal{O} \psi_r^\mathcal{O} f_{\mathbf{i}}^\mathcal{O} &= \begin{cases} (\psi_{r+1}^\mathcal{O} \psi_r^\mathcal{O} \psi_{r+1}^\mathcal{O} - t^{1+\rho_r(\mathbf{i})}) f_{\mathbf{i}}^\mathcal{O}, & \text{if } i_r = i_{r+2} \rightarrow i_{r+1}, \text{ and } \mathbf{i} \in I_+^\gamma, \\ (\psi_{r+1}^\mathcal{O} \psi_r^\mathcal{O} \psi_{r+1}^\mathcal{O} + t^{1-\rho_r(\mathbf{i})}) f_{\mathbf{i}}^\mathcal{O}, & \text{if } i_r = i_{r+2} \leftarrow i_{r+1} \text{ and } \mathbf{i} \in I_-^\gamma, \\ (\psi_{r+1}^\mathcal{O} \psi_r^\mathcal{O} \psi_{r+1}^\mathcal{O} + 1) f_{\mathbf{i}}^\mathcal{O}, & \text{if } i_r = i_{r+2} \leftarrow i_{r+1} \text{ and } \mathbf{i} \in I_+^\gamma, \\ (\psi_{r+1}^\mathcal{O} \psi_r^\mathcal{O} \psi_{r+1}^\mathcal{O} - 1) f_{\mathbf{i}}^\mathcal{O}, & \text{if } i_r = i_{r+2} \rightarrow i_{r+1}, \text{ and } \mathbf{i} \in I_-^\gamma, \\ \psi_{r+1}^\mathcal{O} \psi_r^\mathcal{O} \psi_{r+1}^\mathcal{O} f_{\mathbf{i}}^\mathcal{O}, & \text{otherwise,} \end{cases} \end{aligned}$$

for all admissible  $\mathbf{i}, \mathbf{j} \in I^\gamma$  and  $r, s, t$  satisfying  $2 < r, s < n$  and  $1 \leq t \leq n$ .

*Proof.* First notice that if  $\mathbf{i} \notin I_+^\gamma \cup I_-^\gamma$  then  $f_{\mathbf{i}}^\mathcal{O} = 0$  by [Lemma 2.26](#), so all of the relations above are trivially true. We may assume then that  $\mathbf{i} \in I_+^\gamma \cup I_-^\gamma$ .

The first three identities follow directly from [Theorem 2.19](#) and [Corollary 2.22](#). For the remaining formulas, observe that if  $2 < r < n$  then  $\mathbf{i} \in I_+^\gamma$  if and only if  $s_r \cdot \mathbf{i} \in I_+^\gamma$  and, similarly,  $\mathbf{i} \in I_-^\gamma$  if and only if  $s_r \cdot \mathbf{i} \in I_-^\gamma$ . Therefore, if  $\mathbf{i} \in I_+^\gamma$  the relations

hold by virtue of [Theorem 2.19](#) and if  $\mathbf{i} \in I_-^\gamma$  then they hold by [Corollary 2.22](#). Note that if  $\mathbf{i} \in I_-^\gamma$  then there is a sign change in the last two relations, in comparison with [Corollary 2.22](#), because  $\psi_r^\mathcal{O} f_{\mathbf{i}}^\mathcal{O} = -\psi_r^- f_{\mathbf{i}}^\mathcal{O}$  and  $y_t^\mathcal{O} f_{\mathbf{i}}^\mathcal{O} = -y_s^- f_{\mathbf{i}}^\mathcal{O}$ .  $\square$

Next we need analogues of the relations in [Lemma 2.33](#) for  $\psi_1^\mathcal{O}$ . We could replace the next result with the single relation  $\psi_1^\mathcal{O} = 0$ , however, this is not sufficient for our later arguments because the proof of [Theorem 2.45](#) relies on the fact that the generators of [Proposition 2.31](#) satisfy relations that are compatible with [Definition 1.5](#).

**2.34. Lemma.** *Suppose that  $\gamma \in Q_n^\varepsilon$ . The following identities hold in  $\mathcal{H}_\gamma^\mathcal{O}$ :*

$$\begin{aligned} \psi_1^\mathcal{O} f_{\mathbf{i}}^\mathcal{O} &= f_{s_1, \mathbf{i}}^\mathcal{O} \psi_1^\mathcal{O}, & \psi_1^\mathcal{O} y_s^\mathcal{O} &= y_s^\mathcal{O} \psi_1^\mathcal{O}, & \psi_1^\mathcal{O} \psi_r^\mathcal{O} &= \psi_r^\mathcal{O} \psi_1^\mathcal{O}, \\ \psi_1^\mathcal{O} y_2^\mathcal{O} f_{\mathbf{i}}^\mathcal{O} &= (y_1^\mathcal{O} \psi_1^\mathcal{O} + \delta_{i_1 i_2}) f_{\mathbf{i}}^\mathcal{O}, & y_2^\mathcal{O} \psi_1^\mathcal{O} f_{\mathbf{i}}^\mathcal{O} &= (\psi_1^\mathcal{O} y_1^\mathcal{O} + \delta_{i_1 i_2}) f_{\mathbf{i}}^\mathcal{O}, \\ (\psi_1^\mathcal{O})^2 f_{\mathbf{i}}^\mathcal{O} &= \begin{cases} (y_1^\mathcal{O} - y_2^\mathcal{O}) f_{\mathbf{i}}^\mathcal{O}, & \text{if } i_1 \rightarrow i_2, \\ (y_2^\mathcal{O} - y_1^\mathcal{O}) f_{\mathbf{i}}^\mathcal{O}, & \text{if } i_1 \leftarrow i_2, \\ 0, & \text{if } i_1 = i_2, \\ f_{\mathbf{i}}^\mathcal{O}, & \text{otherwise,} \end{cases} \\ \psi_1^\mathcal{O} \psi_2^\mathcal{O} \psi_1^\mathcal{O} f_{\mathbf{i}}^\mathcal{O} &= \begin{cases} (\psi_2^\mathcal{O} \psi_1^\mathcal{O} \psi_2^\mathcal{O} - 1) f_{\mathbf{i}}^\mathcal{O}, & \text{if } i_1 = i_3 \rightarrow i_2, \\ (\psi_2^\mathcal{O} \psi_1^\mathcal{O} \psi_2^\mathcal{O} + 1) f_{\mathbf{i}}^\mathcal{O}, & \text{if } i_1 = i_3 \leftarrow i_2, \\ \psi_2^\mathcal{O} \psi_1^\mathcal{O} \psi_2^\mathcal{O} f_{\mathbf{i}}^\mathcal{O}, & \text{otherwise,} \end{cases} \end{aligned}$$

for all admissible  $\mathbf{i} \in I^\gamma$  and  $r, s$  satisfying  $2 < r < n$  and  $3 \leq s \leq n$ .

*Proof.* By definition,  $f_{\mathbf{i}}^\mathcal{O} \neq 0$  if and only if  $\mathbf{i} = \text{res}(\mathbf{s})$  for some standard tableau  $\mathbf{s}$ . In particular, if  $\mathbf{i} = (i_1, \dots, i_n)$  then  $i_1 = 0$ ,  $i_2 \in \{-1, 1\}$  and  $i_3 \in \{-2, -1, 1, 2\}$ . Hence, it follows from [Theorem 2.19](#) and [Corollary 2.22](#) that  $\psi_1^+ = 0 = \psi_1^-$ . Therefore,  $\psi_1^\mathcal{O} = 0$  and the first three relations are trivially true. The next two relations hold because  $\delta_{i_1 i_2} = 0$  whenever  $f_{\mathbf{i}}^\mathcal{O} \neq 0$  and the quadratic relation for  $(\psi_1^\mathcal{O})^2$  holds in view of [Theorem 2.19](#) and [Corollary 2.22](#). For the final “braid” relation, if  $i_1 = i_3 \rightarrow i_2$  or  $i_1 = i_3 \leftarrow i_2$  then  $f_{\mathbf{i}}^\mathcal{O} = 0$  by the remarks at the start of the proof, so the braid relation is trivially true in these cases. In the remaining cases  $\psi_1^\mathcal{O} \psi_2^\mathcal{O} \psi_1^\mathcal{O} = 0 = \psi_2^\mathcal{O} \psi_1^\mathcal{O} \psi_2^\mathcal{O}$  since  $\psi_1^\mathcal{O} = 0$ . This completes the proof.  $\square$

It remains to determine the relations involving  $\psi_2^\mathcal{O}$ . The first step is easy.

**2.35. Lemma.** *Suppose that  $\mathbf{i} \in I^\gamma$ ,  $2 < r < n$  and  $1 \leq s \leq n$  with  $t \neq 2, 3$ . Then*

$$\psi_2^\mathcal{O} \psi_r^\mathcal{O} = \psi_r^\mathcal{O} \psi_2^\mathcal{O} \quad \text{and} \quad \psi_2^\mathcal{O} y_s^\mathcal{O} = y_s^\mathcal{O} \psi_2^\mathcal{O}.$$

*Proof.* Since  $\psi_2^\mathcal{O} f_{\mathbf{i}}^\mathcal{O} = \pm \kappa_{\mathbf{i}} \psi_2^\pm f_{\mathbf{i}}^\mathcal{O}$ , where  $\kappa_{\mathbf{i}} \in \mathcal{O}$  is invertible for  $\mathbf{i} \in I^\gamma$ , the result follows directly from [Theorem 2.19](#) and [Corollary 2.22](#).  $\square$

For the remaining relations we need a more precise description of how the generators of [Definition 2.29](#) act on the seminormal basis. Suppose that  $\mathbf{s} \in \text{Std}(\mathbf{i})$  and that  $\mathbf{u} = (r, r+1)\mathbf{s}$ , where  $1 \leq r < n$ . Following [\[9, \(4.21\)\]](#), define

$$(2.36) \quad \beta_r(\mathbf{s}) = \begin{cases} -\beta_r(\mathbf{s}'), & \text{if } \mathbf{i} \in I_-^\gamma, \\ \frac{t^{\hat{i}_r - c_r(\mathbf{s})} \alpha_r(\mathbf{s})}{[1 - \rho_r(\mathbf{s})]}, & \text{if } \mathbf{i} \in I_+^\gamma \text{ and } i_r = i_{r+1}, \\ \frac{t^{c_{r+1}(\mathbf{s}) - \hat{i}_r} \alpha_r(\mathbf{s}) [\rho_r(\mathbf{s})]}{t^{-\rho_r(\mathbf{s})} \alpha_r(\mathbf{s}) [\rho_r(\mathbf{s})]}, & \text{if } \mathbf{i} \in I_+^\gamma \text{ and } i_r = i_{r+1} + 1, \\ \frac{t^{-\rho_r(\mathbf{s})} \alpha_r(\mathbf{s}) [\rho_r(\mathbf{s})]}{[1 - \rho_r(\mathbf{s})]}, & \text{if } \mathbf{i} \in I_+^\gamma \text{ and } i_r \notin \{i_{r+1}, i_{r+1} + 1\}. \end{cases}$$

Note that  $\beta_r(\mathbf{s}) = 0$  if  $\mathbf{u}$  is not standard because  $\alpha_r(\mathbf{u}) = 0$  whenever  $\mathbf{u} \notin \text{Std}(\mathcal{P}_n)$ . More explicit formulas for  $\beta_r(\mathbf{s})$  can be obtained using (2.28), however, we will only need these in one special case; see (2.40) below.

Note that if  $\mathbf{s}$  is standard and  $\mathbf{i} = \text{res}(\mathbf{s})$  then  $\text{res}(\mathbf{s}') = -\mathbf{i}$ . Therefore, the four cases in (2.36) are mutually exclusive. We need to be slightly careful, however, because if  $\mathbf{i} = \text{res}(\mathbf{s})$  and  $\mathbf{j} = \text{res}(\mathbf{s}')$  then it is not usually true that  $\hat{j}_r = -\hat{i}_r$ , for  $1 \leq r \leq n$ .

Following [9, Lemma 4.23] we can now describe the action of  $\psi_r^\mathcal{O}$  and  $y_r^\mathcal{O}$  on the seminormal basis  $\{f_{\mathbf{st}}\}$ . This result is the only place where we explicitly use the assumption of Definition 2.27.

**2.37. Proposition.** *Suppose that  $\mathbf{s}, \mathbf{t} \in \text{Std}(\lambda)$  for  $\lambda \in \mathcal{P}_n$  and let  $\mathbf{i} = \text{res}(\mathbf{s})$ ,  $\mathbf{j} = \text{res}(\mathbf{s}')$  for  $\mathbf{i}, \mathbf{j} \in I^\gamma$ . Fix  $1 \leq r < n$  and let  $\mathbf{u} = (r, r+1)\mathbf{s}$ . Then*

$$\psi_r^\mathcal{O} f_{\mathbf{st}} = \begin{cases} \kappa_{\mathbf{i}} \beta_2(\mathbf{s}) f_{\mathbf{ut}}, & \text{if } r = 2, \\ \beta_r(\mathbf{s}) f_{\mathbf{ut}} - \delta_{i_r, i_{r+1}} \frac{t^{\hat{i}_{r+1} - c_{r+1}(\mathbf{s})}}{[\rho_r(\mathbf{s})]} f_{\mathbf{st}}, & \text{if } r \neq 2 \text{ and } \mathbf{i} \in I_+^\gamma, \\ \beta_r(\mathbf{s}) f_{\mathbf{ut}} - \delta_{i_r, i_{r+1}} \frac{t^{\hat{j}_{r+1} - c_{r+1}(\mathbf{s})}}{[\rho_r(\mathbf{s})]} f_{\mathbf{st}}, & \text{if } r \neq 2 \text{ and } \mathbf{i} \in I_-^\gamma. \end{cases}$$

Moreover, if  $1 \leq k \leq n$  then

$$y_k^\mathcal{O} f_{\mathbf{st}} = \begin{cases} [c_k(\mathbf{s}) - \hat{i}_k] f_{\mathbf{st}}, & \text{if } \mathbf{i} \in I_+^\gamma, \\ -[c_k(\mathbf{s}') - \hat{j}_k] f_{\mathbf{st}}, & \text{if } \mathbf{i} \in I_-^\gamma. \end{cases}$$

*Proof.* Without loss of generality, we can assume that  $\mathbf{t} = \mathbf{s}$  by Theorem 2.4(a), so we need to compute  $\psi_r^\mathcal{O} f_{\mathbf{ss}}$  and  $y_r^\mathcal{O} f_{\mathbf{ss}}$ .

First consider  $\psi_r^\mathcal{O} f_{\mathbf{ss}}$  when  $r \neq 2$ . If  $\mathbf{i} \in I_+^\gamma$  then  $\psi_r^\mathcal{O} f_{\mathbf{ss}} = \psi_r^+ f_{\mathbf{ss}}$  and the lemma is a restatement of [9, Lemma 4.23]. Suppose then that  $\mathbf{i} \in I_-^\gamma$ , so that  $\mathbf{j} \in I_+^\gamma$  and  $\psi_r^\mathcal{O} f_{\mathbf{s}'\mathbf{s}'}$  is given by the formulas above. As  $\#$  is an involution, using Corollary 2.8 for the third equality and Lemma 2.9 for the last equality,

$$\begin{aligned} \psi_r^\mathcal{O} f_{\mathbf{ss}} &= -(\psi_r^+)^{\#} f_{\mathbf{ss}} = -(\psi_r^+ f_{\mathbf{ss}})^{\#} = -\frac{\gamma_{\mathbf{s}}}{\gamma_{\mathbf{s}'}} (\psi_r^+ f_{\mathbf{s}'\mathbf{s}'})^{\#}, \\ &= -\frac{\gamma_{\mathbf{s}}}{\gamma_{\mathbf{s}'}} \left( \beta_r(\mathbf{s}') f_{\mathbf{u}'\mathbf{s}'} - \delta_{j_r, j_{r+1}} \frac{t^{\hat{j}_{r+1} - c_{r+1}(\mathbf{s}')}}{[\rho_r(\mathbf{s}')] } f_{\mathbf{s}'\mathbf{s}'} \right)^{\#} \\ &= \frac{\alpha_r(\mathbf{s}) \beta_r(\mathbf{s}')}{\alpha_r(\mathbf{s}')} f_{\mathbf{us}} - \delta_{i_r, i_{r+1}} \frac{t^{\hat{j}_{r+1} - c_{r+1}(\mathbf{s})}}{[\rho_r(\mathbf{s})]} f_{\mathbf{ss}}, \end{aligned}$$

since  $[\rho_r(\mathbf{s}')] = [-\rho_r(\mathbf{s})] = -t^{-\rho_r(\mathbf{s})} [\rho_r(\mathbf{s})]$ . By Definition 2.27,  $\alpha_r(\mathbf{s}') = -\alpha_r(\mathbf{s})$  and  $\beta_r(\mathbf{s}') = -\beta_r(\mathbf{s})$ , so this establishes the formula for  $\psi_r^\mathcal{O} f_{\mathbf{st}}$  when  $r \neq 2$ .

Now consider  $\psi_2^\mathcal{O} f_{\mathbf{ss}}$ . If  $f_{\mathbf{i}}^\mathcal{O} \neq 0$  then  $i_2 \neq i_3$  because  $f_{\mathbf{i}}^\mathcal{O} \neq 0$  only if  $\mathbf{i}$  is the residue sequence of some standard tableau. Therefore, if  $\mathbf{i} \in I_+^\gamma$  and  $f_{\mathbf{i}}^\mathcal{O} \neq 0$  then the argument of the last paragraph shows that  $\psi_2^+ f_{\mathbf{st}} = \beta_2(\mathbf{s}) f_{\mathbf{ut}}$  and if  $\mathbf{i} \in I_-^\gamma$  then  $-\psi_2^- f_{\mathbf{st}} = \beta_2(\mathbf{s}) f_{\mathbf{ut}}$ . As  $\psi_2^\mathcal{O} = \sum_{\mathbf{i} \in I_+^\gamma} \kappa_{\mathbf{i}} (\psi_2^+ f_{\mathbf{i}}^\mathcal{O} - \psi_2^- f_{-\mathbf{i}}^\mathcal{O})$ , it follows that  $\psi_2^\mathcal{O} f_{\mathbf{st}} = \kappa_{\mathbf{i}} \beta_2(\mathbf{s}) f_{\mathbf{ut}}$  as claimed.

For the action of  $y_k^\mathcal{O}$ , if  $\mathbf{i} \in I_+^\gamma$  then  $y_k^\mathcal{O} f_{\mathbf{ss}} = y_k^+ f_{\mathbf{ss}} = [c_k(\mathbf{s}) - \hat{i}_k] f_{\mathbf{ss}}$  by [9, Lemma 4.23]. On the other hand, if  $\mathbf{i} \in I_-^\gamma$  then, using Lemma 2.12 twice,

$$y_k^\mathcal{O} f_{\mathbf{ss}} = -\frac{\gamma_{\mathbf{s}}}{\gamma_{\mathbf{s}'}} (y_k^+ f_{\mathbf{s}'\mathbf{s}'})^{\#} = -\frac{\gamma_{\mathbf{s}}}{\gamma_{\mathbf{s}'}} ([c_k(\mathbf{s}') - \hat{j}_k] f_{\mathbf{s}'\mathbf{s}'})^{\#} = -[c_k(\mathbf{s}') - \hat{j}_k] f_{\mathbf{ss}}.$$

as required.  $\square$

We can now determine the remaining “KLR-like” relations satisfied by  $\psi_2^\mathcal{O}$ .

**2.38. Lemma.** *Suppose that  $\mathbf{i} \in I^\gamma$  and let  $\mathbf{j} = s_2 \cdot \mathbf{i}$ . Then  $\psi_2^\mathcal{O} f_{\mathbf{i}}^\mathcal{O} = f_{\mathbf{j}}^\mathcal{O} \psi_2^\mathcal{O}$ .*

*Proof.* If  $\mathbf{i} \notin I_+^\gamma \cup I_-^\gamma$  then  $f_{\mathbf{i}}^\mathcal{O} = 0 = f_{\mathbf{j}}^\mathcal{O}$  and there is nothing to prove. Therefore, we may assume that  $\mathbf{i} \in I_+^\gamma \cup I_-^\gamma$ . Recall that  $f_\gamma^\mathcal{O} = \sum_{\mathbf{k} \in I^\gamma} f_{\mathbf{k}}^\mathcal{O}$  is the identity element of  $\mathcal{H}_\gamma^\mathcal{O}$ . By [Proposition 2.37](#),

$$\psi_2^\mathcal{O} = \psi_2^\mathcal{O} f_\gamma^\mathcal{O} = \sum_{\mathbf{k} \in I^\gamma} \psi_2^\mathcal{O} f_{\mathbf{k}}^\mathcal{O} = \sum_{\substack{\mathbf{k} \in I^\gamma \\ \mathbf{t} \in \text{Std}(\mathbf{k})}} \frac{1}{\gamma_{\mathbf{t}}} \psi_2^\mathcal{O} f_{\mathbf{t}\mathbf{t}} = \sum_{\substack{\mathbf{k} \in I^\gamma \\ \mathbf{t} \in \text{Std}(\mathbf{k}) \\ \mathbf{s} = s_2 \mathbf{t}}} \frac{\kappa_{\mathbf{k}} \beta_2(\mathbf{t})}{\gamma_{\mathbf{t}}} f_{\mathbf{st}}.$$

If  $f_{\mathbf{st}}$  is a term in the right-hand sum, with  $\mathbf{t} \in \text{Std}(\mathbf{k})$  and  $\mathbf{s} = s_2 \mathbf{t}$ , then

$$f_{\mathbf{st}} f_{\mathbf{i}}^\mathcal{O} = \delta_{\mathbf{i}\mathbf{k}} f_{\mathbf{st}} = \delta_{s_2 \cdot \mathbf{i}, s_2 \cdot \mathbf{k}} f_{\mathbf{st}} = f_{\mathbf{j}}^\mathcal{O} f_{\mathbf{st}},$$

by [Theorem 2.4\(b\)](#). Hence,  $\psi_2^\mathcal{O} f_{\mathbf{i}}^\mathcal{O} = \sum_{\mathbf{t} \in \text{Std}(\mathbf{i})} \frac{1}{\gamma_{\mathbf{t}}} \kappa_{\mathbf{i}} \beta_2(\mathbf{t}) f_{\mathbf{st}} = f_{\mathbf{j}}^\mathcal{O} \psi_2^\mathcal{O}$ , as required.  $\square$

Recall from [\(2.32\)](#) that if  $d \in \mathbb{Z}$  and  $\mathbf{i} \in I_\pm^\gamma$  then  $y_r^{\langle d \rangle} f_{\mathbf{i}}^\mathcal{O} = (t^d y_r^\mathcal{O} \mp [d]) f_{\mathbf{i}}^\mathcal{O}$ .

**2.39. Lemma.** *Suppose that  $\gamma \in Q_n^\varepsilon$  and  $\mathbf{i} \in I^\gamma$ . Then:*

$$y_3^\mathcal{O} \psi_2^\mathcal{O} f_{\mathbf{i}}^\mathcal{O} = (\psi_2^\mathcal{O} y_2^{\langle -e \rangle} + \delta_{i_2 i_3}) f_{\mathbf{i}}^\mathcal{O} \quad \text{and} \quad \psi_2^\mathcal{O} y_3^\mathcal{O} f_{\mathbf{i}}^\mathcal{O} = (y_2^{\langle -e \rangle} \psi_2^\mathcal{O} + \delta_{i_2 i_3}) f_{\mathbf{i}}^\mathcal{O}.$$

*Proof.* Both identities are proved similarly so we consider only the first one. If  $f_{\mathbf{i}}^\mathcal{O} \neq 0$  then  $\mathbf{i} = \text{res}(\mathbf{s})$ , for some standard tableau  $\mathbf{s}$ , in which case  $i_2 \neq i_3$ . Hence, if  $f_{\mathbf{i}}^\mathcal{O} \neq 0$  then  $\delta_{i_2 i_3} = 0$  so we can assume that  $\delta_{i_2 i_3} = 0$  in what follows. (We include the term for  $\delta_{i_2 i_3}$  because to prove [Theorem A](#) we need to compare the identity in the lemma with the relations in [Definition 1.5](#).) Without loss of generality, we may assume that  $\mathbf{i} \in I_+^\gamma$ .

By [Theorem 2.4\(b\)](#),  $f_{\mathbf{i}}^\mathcal{O} = \sum_{\mathbf{s} \in \text{Std}(\mathbf{i})} \frac{1}{\gamma_{\mathbf{s}}} f_{\mathbf{ss}}$ . Therefore, to prove the lemma it is enough to verify that  $y_3^\mathcal{O} \psi_2^\mathcal{O} f_{\mathbf{ss}} = \psi_2^\mathcal{O} y_2^{\langle -e \rangle} f_{\mathbf{ss}}$ , for all  $\mathbf{s} \in \text{Std}(\mathbf{i})$ . Fix  $\mathbf{s} \in \text{Std}(\mathbf{i})$  and set  $\mathbf{u} = (2, 3)\mathbf{s}$  and  $\mathbf{j} = s_2 \cdot \mathbf{i} \in I_-^\gamma$  so that  $\mathbf{j} = \text{res}(\mathbf{u})$  if  $\mathbf{u}$  is standard. If  $\mathbf{u}$  is not standard then  $\beta_2(\mathbf{s}) = 0$  so  $y_3^\mathcal{O} \psi_2^\mathcal{O} f_{\mathbf{ss}} = 0 = \psi_2^\mathcal{O} y_2^{\langle -e \rangle} f_{\mathbf{ss}}$  by [Proposition 2.37](#). Suppose then that  $\mathbf{u}$  is standard so that

$$\mathbf{s}_{\downarrow 3} = \begin{bmatrix} 1 & 2 \\ 3 \end{bmatrix} \quad \text{and} \quad \mathbf{u}_{\downarrow 3} = \begin{bmatrix} 1 & 3 \\ 2 \end{bmatrix}.$$

Using [Proposition 2.37](#) again,  $\psi_2^\mathcal{O} y_2^{\langle -e \rangle} f_{\mathbf{ss}} = -\kappa_{\mathbf{i}} \beta_2(\mathbf{s}) [-e] f_{\mathbf{us}}$  since  $y_2^\mathcal{O} f_{\mathbf{ss}} = 0$ . Similarly,  $y_3^\mathcal{O} \psi_2^\mathcal{O} f_{\mathbf{ss}} = \kappa_{\mathbf{i}} \beta_2(\mathbf{s}) y_3^\mathcal{O} f_{\mathbf{us}} = -\kappa_{\mathbf{i}} \beta_2(\mathbf{s}) [-e] f_{\mathbf{us}}$ . Hence,  $y_3^\mathcal{O} \psi_2^\mathcal{O} f_{\mathbf{ss}} = \psi_2^\mathcal{O} y_2^{\langle -e \rangle} f_{\mathbf{ss}}$  in all cases, completing the proof.  $\square$

The proof of the next result explains why  $\kappa_{\mathbf{i}}$  is needed in the definition of  $\psi_2^\mathcal{O}$ . Fix  $\mathbf{s} \in \text{Std}(\mathbf{i})$  such that  $\mathbf{i} \in I_+^\gamma$  and  $(2, 3)\mathbf{s}$  is standard. Then  $\rho_2(\mathbf{s}) = 2$  and either  $e = 3$  and  $i_2 \rightarrow i_3$ , or  $e > 3$  and  $i_2 \not\rightarrow i_3$ . Hence,  $i_2 \notin \{i_3, i_3 + 1\}$ , so by [\(2.36\)](#) and [Definition 2.29](#)

$$(2.40) \quad \beta_2(\mathbf{s}) = \frac{t^{-\rho_2(\mathbf{s})} \sqrt{-1} \sqrt{t} \sqrt{[3]} [\rho_2(\mathbf{s})]}{[1 - \rho_2(\mathbf{s})][2]} = -\frac{\sqrt{-1} \sqrt{[3]}}{\sqrt{t}}.$$

We can now determine the quadratic relation for  $\psi_2^\mathcal{O}$ .

**2.41. Lemma.** *Suppose that  $\mathbf{i} \in I^\gamma$ . Then*

$$(\psi_2^\mathcal{O})^2 f_{\mathbf{i}}^\mathcal{O} = \begin{cases} (y_2^\mathcal{O} - y_3^\mathcal{O}) f_{\mathbf{i}}^\mathcal{O}, & \text{if } i_2 \rightarrow i_3, \\ (y_3^\mathcal{O} - y_2^\mathcal{O}) f_{\mathbf{i}}^\mathcal{O}, & \text{if } i_2 \leftarrow i_3, \\ 0, & \text{if } i_2 = i_3, \\ f_{\mathbf{i}}^\mathcal{O}, & \text{otherwise.} \end{cases}$$

*Proof.* It is enough to consider the case when  $\mathbf{i} \in I_+^\gamma$ . Since  $f_{\mathbf{i}}^\mathcal{O} = \sum_{\mathbf{s} \in \text{Std}(\mathbf{i})} \frac{1}{\gamma_{\mathbf{s}}} f_{\mathbf{s}\mathbf{s}}$  we are reduced to computing  $(\psi_2^\mathcal{O})^2 f_{\mathbf{s}\mathbf{s}}$ , for  $\mathbf{s} \in \text{Std}(\mathbf{i})$  and  $\mathbf{i} \in I_+^\gamma$ . Fix  $\mathbf{s} \in \text{Std}(\mathbf{i})$  and let  $\mathbf{u} = (2, 3)\mathbf{s} \in \text{Std}(\mathbf{j})$ . By [Proposition 2.37](#),

$$(\psi_2^\mathcal{O})^2 f_{\mathbf{s}\mathbf{s}} = \kappa_{\mathbf{i}} \beta_2(\mathbf{s}) \psi_2^\mathcal{O} f_{\mathbf{u}\mathbf{s}} = -\kappa_{\mathbf{i}}^2 \beta_2(\mathbf{s})^2 f_{\mathbf{s}\mathbf{s}}.$$

If 3 is in the first row of  $\mathbf{s}$  then  $\mathbf{u}$  is not standard so  $\beta_2(\mathbf{s}) = 0$  and  $(\psi_2^\mathcal{O})^2 f_{\mathbf{s}\mathbf{s}} = 0$ . In this case,  $i_2 \rightarrow i_3$  and  $y_2^\mathcal{O} f_{\mathbf{s}\mathbf{s}} = 0 = y_3^\mathcal{O} f_{\mathbf{s}\mathbf{s}}$ , so the lemma holds. The only other possibility is that 3 is in the first column of  $\mathbf{s}$ , so that  $\rho_2(\mathbf{s}) = 2$ . Then  $i_2 \rightarrow i_3$  if  $e = 3$  and  $i_2 \not\rightarrow i_3$  if  $e > 3$ . Hence, using [Definition 2.29](#) and [\(2.40\)](#),

$$(\psi_2^\mathcal{O})^2 f_{\mathbf{s}\mathbf{s}} = -\kappa_{\mathbf{i}}^2 \beta_2(\mathbf{s})^2 f_{\mathbf{s}\mathbf{s}} = \begin{cases} t^{-3}[3] f_{\mathbf{s}\mathbf{s}}, & \text{if } i_2 \rightarrow i_3 \text{ (and } e = 3), \\ f_{\mathbf{s}\mathbf{s}}, & \text{if } i_2 \not\rightarrow i_3 \text{ (and } e > 3). \end{cases}$$

Hence, if  $i_2 \not\rightarrow i_3$  then  $(\psi_2^\mathcal{O})^2 f_{\mathbf{i}}^\mathcal{O} = f_{\mathbf{i}}^\mathcal{O}$  as claimed. Finally, if  $i_2 \rightarrow i_3$  then

$$(y_2^\mathcal{O} - y_3^\mathcal{O}) f_{\mathbf{s}\mathbf{s}} = (0 - [-e]) f_{\mathbf{s}\mathbf{s}} = t^{-3}[3] f_{\mathbf{s}\mathbf{s}} = (\psi_2^\mathcal{O})^2 f_{\mathbf{s}\mathbf{s}},$$

where the middle equality holds only because  $e = 3$ . This completes the proof.  $\square$

*Remark.* The proof of [Lemma 2.41](#) suggests that  $\kappa_{\mathbf{i}}$  is uniquely determined, for  $\mathbf{i} \in I^\gamma$ . In fact, this is not quite true. What the proof shows is that the value of  $\kappa_{\mathbf{i}}$  is uniquely determined by the quadratic relation satisfied by  $\psi_2^\mathcal{O}$ . For the proof of our main results we only need  $\psi_2^\mathcal{O}$  to satisfy a “deformed” version of the quadratic relation for  $\psi_2$  in [Lemma 2.41](#). For example, we can obtain slightly different relations by replacing  $y_r^\mathcal{O}$  with  $y_r^{\langle ke \rangle}$ , for some  $k \in \mathbb{Z}$ . Such relations would require a different value for  $\kappa_{\mathbf{i}}$ .

Finally, it remains to check the braid relation for  $\psi_2^\mathcal{O}$  and  $\psi_3^\mathcal{O}$ .

**2.42. Lemma.** *Suppose that  $\mathbf{i} \in I^\gamma$ . Then*

$$\psi_2^\mathcal{O} \psi_3^\mathcal{O} \psi_2^\mathcal{O} f_{\mathbf{i}}^\mathcal{O} = \begin{cases} (\psi_3^\mathcal{O} \psi_2^\mathcal{O} \psi_3^\mathcal{O} - 1) f_{\mathbf{i}}^\mathcal{O}, & \text{if } i_2 = i_4 \rightarrow i_3, \\ (\psi_3^\mathcal{O} \psi_2^\mathcal{O} \psi_3^\mathcal{O} + 1) f_{\mathbf{i}}^\mathcal{O}, & \text{if } i_2 = i_4 \leftarrow i_3, \\ \psi_3^\mathcal{O} \psi_2^\mathcal{O} \psi_3^\mathcal{O} f_{\mathbf{i}}^\mathcal{O}, & \text{otherwise,} \end{cases}$$

*Proof.* Again, it is enough to consider the case when  $\mathbf{i} \in I_+^\gamma$ . We fix  $\mathbf{s} \in \text{Std}(\mathbf{i})$  and show that the two sides of the identity in the lemma act in the same way on  $f_{\mathbf{s}\mathbf{s}}$ . Let  $\mathbf{j} = (2, 4) \cdot \mathbf{i} \in I^\gamma$ . By [Lemma 2.38](#) and [Lemma 2.35](#),  $\psi_2^\mathcal{O} \psi_3^\mathcal{O} \psi_2^\mathcal{O} f_{\mathbf{i}}^\mathcal{O} = f_{\mathbf{j}}^\mathcal{O} \psi_2^\mathcal{O} \psi_3^\mathcal{O} \psi_2^\mathcal{O}$ , so  $\psi_2^\mathcal{O} \psi_3^\mathcal{O} \psi_2^\mathcal{O} f_{\mathbf{i}}^\mathcal{O} = 0$  unless  $\mathbf{j}$  is the residue sequence of a standard tableau. Similarly  $\psi_3^\mathcal{O} \psi_2^\mathcal{O} \psi_3^\mathcal{O} f_{\mathbf{i}}^\mathcal{O} = 0$  unless  $\mathbf{j}$  is the residue sequence of a standard tableau. Let  $\mathbf{s}_{\downarrow 4}$  be the subtableau of  $\mathbf{s}$  containing the numbers 1, 2, 3, 4. Then

$$\mathbf{s}_{\downarrow 4} \in \left\{ \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 \\ \hline 4 \\ \hline \end{array} \right\}$$

since  $\mathbf{i} \in I_+^\gamma$ . We consider two cases.

**Case 1:  $e > 3$ :** Inspecting the list of possibilities for  $s_{\downarrow 4}$ , in all cases  $i_2 \neq i_4$  and  $\mathbf{j} \neq \text{res}(\mathbf{t})$  for any standard tableau  $\mathbf{t}$ . Therefore,

$$\psi_2^\mathcal{O} \psi_3^\mathcal{O} \psi_2^\mathcal{O} f_{\mathbf{i}}^\mathcal{O} = 0 = \psi_3^\mathcal{O} \psi_2^\mathcal{O} \psi_3^\mathcal{O} f_{\mathbf{i}}^\mathcal{O},$$

in agreement with the statement of the lemma.

**Case 2:  $e = 3$ :** Except for the last tableau in the set above,  $i_2 \neq i_4$  and  $\mathbf{j}$  is not a residue sequence for a standard tableau. Hence, as in Case 1, the lemma holds when  $i_2 \neq i_4$  as both sides are zero. Moreover, if  $f_{\mathbf{i}}^\mathcal{O} \neq 0$  then the case  $i_2 = i_4 \leftarrow i_3$  does not arise, so the lemma is vacuously true in this case. It remains to consider the case when  $i_2 = i_4 \rightarrow i_3$ , which occurs only if  $s_{\downarrow 4}$  is the last tableau in the set above. Noting that  $\psi_3^\mathcal{O} f_{\text{ss}} = 0$ , [Proposition 2.37](#) and [\(2.40\)](#) quickly imply that

$$(\psi_2^\mathcal{O} \psi_3^\mathcal{O} \psi_2^\mathcal{O} - \psi_3^\mathcal{O} \psi_2^\mathcal{O} \psi_3^\mathcal{O}) f_{\text{ss}} = -\kappa_{\mathbf{i}}^2 \beta_2(s)^2 \frac{t^3}{[3]} f_{\text{ss}} = -f_{\text{ss}},$$

where the last equality follows using [\(2.40\)](#) exactly as in the proof of [Lemma 2.41](#). This completes the proof.  $\square$

**2.5. The isomorphism  $\mathcal{R}_e(\mathfrak{A}_n) \cong \mathcal{H}_\xi(\mathfrak{A}_n)$ .** We now have almost everything in place that we need to prove [Theorem A](#). We first prove a stronger version of [Theorem 1.9](#) over  $\mathcal{O}$ . For this we need the following definition, which should be viewed as an  $\mathcal{O}$ -deformation of  $\mathcal{R}_e(\mathfrak{S}_n)$ . The reader should compare this result with [Theorem 2.19](#).

**2.43. Definition.** Suppose that  $\gamma \in Q_n^\varepsilon$ . Let  $\hat{R}_\gamma^\mathcal{O}$  be the unital associative  $\mathcal{O}$ -algebra generated by the elements

$$\{\psi_1^\mathcal{O}, \psi_2^\mathcal{O}, \dots, \psi_{n-1}^\mathcal{O}\} \cup \{\dot{y}_1^\mathcal{O}, \dot{y}_2^\mathcal{O}, \dots, \dot{y}_n^\mathcal{O}\} \cup \{f_{\mathbf{i}}^\mathcal{O} \mid \mathbf{i} \in I^\gamma\}$$

subject to the relations

$$\begin{aligned} (\dot{y}_1^\mathcal{O})^{(\Lambda_0, \alpha_{i_1})} f_{\mathbf{i}}^\mathcal{O} &= 0, & f_{\mathbf{i}}^\mathcal{O} f_{\mathbf{j}}^\mathcal{O} &= \delta_{\mathbf{ij}} f_{\mathbf{i}}^\mathcal{O}, & \sum_{\mathbf{i} \in I^\gamma} f_{\mathbf{i}}^\mathcal{O} &= 1, \\ \dot{y}_t^\mathcal{O} f_{\mathbf{i}}^\mathcal{O} &= f_{\mathbf{i}}^\mathcal{O} \dot{y}_t^\mathcal{O}, & \psi_r^\mathcal{O} f_{\mathbf{i}}^\mathcal{O} &= f_{s_r \cdot \mathbf{i}}^\mathcal{O} \psi_r^\mathcal{O}, & \dot{y}_r^\mathcal{O} \dot{y}_t^\mathcal{O} &= \dot{y}_t^\mathcal{O} \dot{y}_r^\mathcal{O}, \\ \psi_1^\mathcal{O} \dot{y}_2^\mathcal{O} f_{\mathbf{i}}^\mathcal{O} &= (\dot{y}_1^\mathcal{O} \psi_1^\mathcal{O} + \delta_{i_1 i_2}) f_{\mathbf{i}}^\mathcal{O}, & \dot{y}_2^\mathcal{O} \psi_1^\mathcal{O} f_{\mathbf{i}}^\mathcal{O} &= (\psi_1^\mathcal{O} \dot{y}_1^\mathcal{O} + \delta_{i_1 i_2}) f_{\mathbf{i}}^\mathcal{O}, \\ \psi_2^\mathcal{O} \dot{y}_3^\mathcal{O} f_{\mathbf{i}}^\mathcal{O} &= (\dot{y}_2^{(-e)} \psi_2^\mathcal{O} + \delta_{i_2 i_3}) f_{\mathbf{i}}^\mathcal{O}, & \dot{y}_3^\mathcal{O} \psi_2^\mathcal{O} f_{\mathbf{i}}^\mathcal{O} &= (\psi_2^\mathcal{O} \dot{y}_2^{(-e)} + \delta_{i_2 i_3}) f_{\mathbf{i}}^\mathcal{O}, \\ \psi_r^\mathcal{O} \dot{y}_t^\mathcal{O} &= \dot{y}_t^\mathcal{O} \psi_r^\mathcal{O}, & \text{if } t \neq r, r+1, \\ \psi_r^\mathcal{O} \psi_s^\mathcal{O} &= \psi_s^\mathcal{O} \psi_r^\mathcal{O}, & \text{if } |r-s| > 1, \end{aligned}$$

if  $r = 1$  or  $r = 2$  then

$$\begin{aligned} (\psi_r^\mathcal{O})^2 f_{\mathbf{i}}^\mathcal{O} &= \begin{cases} (\dot{y}_r^\mathcal{O} - \dot{y}_{r+1}^\mathcal{O}) f_{\mathbf{i}}^\mathcal{O}, & \text{if } i_r \rightarrow i_{r+1}, \\ (\dot{y}_{r+1}^\mathcal{O} - \dot{y}_r^\mathcal{O}) f_{\mathbf{i}}^\mathcal{O}, & \text{if } i_r \leftarrow i_{r+1}, \\ 0, & \text{if } i_r = i_{r+1}, \\ f_{\mathbf{i}}^\mathcal{O}, & \text{otherwise,} \end{cases} \\ \psi_r^\mathcal{O} \psi_{r+1}^\mathcal{O} \psi_r^\mathcal{O} f_{\mathbf{i}}^\mathcal{O} &= \begin{cases} (\psi_{r+1}^\mathcal{O} \psi_r^\mathcal{O} \psi_{r+1}^\mathcal{O} - 1) f_{\mathbf{i}}^\mathcal{O}, & \text{if } i_r = i_{r+2} \rightarrow i_{r+1}, \\ (\psi_{r+1}^\mathcal{O} \psi_r^\mathcal{O} \psi_{r+1}^\mathcal{O} + 1) f_{\mathbf{i}}^\mathcal{O}, & \text{if } i_r = i_{r+2} \leftarrow i_{r+1}, \\ \psi_{r+1}^\mathcal{O} \psi_r^\mathcal{O} \psi_{r+1}^\mathcal{O} f_{\mathbf{i}}^\mathcal{O}, & \text{otherwise,} \end{cases} \end{aligned}$$

and if  $2 < r < n$  then

$$\dot{y}_r^\mathcal{O} \dot{y}_{r+1}^\mathcal{O} f_{\mathbf{i}}^\mathcal{O} = (\dot{y}_r^\mathcal{O} \dot{y}_r^\mathcal{O} + \delta_{i_r i_{r+1}}) f_{\mathbf{i}}^\mathcal{O}, \quad \dot{y}_{r+1}^\mathcal{O} \dot{y}_r^\mathcal{O} f_{\mathbf{i}}^\mathcal{O} = (\dot{y}_r^\mathcal{O} \dot{y}_r^\mathcal{O} + \delta_{i_r i_{r+1}}) f_{\mathbf{i}}^\mathcal{O},$$

$$(\psi_r^\mathcal{O})^2 f_{\mathbf{i}}^\mathcal{O} = \begin{cases} (\dot{y}_r^{\langle 1+\rho_r(\mathbf{i}) \rangle} - \dot{y}_{r+1}^\mathcal{O}) f_{\mathbf{i}}^\mathcal{O}, & \text{if } i_r \rightarrow i_{r+1} \text{ and } \mathbf{i} \in I_+^\gamma \\ (\dot{y}_{r+1}^\mathcal{O} - \dot{y}_r^{\langle 1-\rho_r(\mathbf{i}) \rangle}) f_{\mathbf{i}}^\mathcal{O}, & \text{if } i_r \leftarrow i_{r+1} \text{ and } \mathbf{i} \in I_-^\gamma \\ (\dot{y}_{r+1}^{\langle 1-\rho_r(\mathbf{i}) \rangle} - \dot{y}_r^\mathcal{O}) f_{\mathbf{i}}^\mathcal{O}, & \text{if } i_r \leftarrow i_{r+1} \text{ and } \mathbf{i} \in I_+^\gamma \\ (\dot{y}_r^\mathcal{O} - \dot{y}_{r+1}^{\langle 1+\rho_r(\mathbf{i}) \rangle}) f_{\mathbf{i}}^\mathcal{O}, & \text{if } i_r \rightarrow i_{r+1} \text{ and } \mathbf{i} \in I_-^\gamma \\ 0, & \text{if } i_r = i_{r+1}, \\ f_{\mathbf{i}}^\mathcal{O}, & \text{otherwise,} \end{cases}$$

$$\psi_r^\mathcal{O} \psi_{r+1}^\mathcal{O} \psi_r^\mathcal{O} f_{\mathbf{i}}^\mathcal{O} = \begin{cases} (\psi_{r+1}^\mathcal{O} \psi_r^\mathcal{O} \psi_{r+1}^\mathcal{O} - t^{1+\rho_r(\mathbf{i})}) f_{\mathbf{i}}^\mathcal{O}, & \text{if } i_r = i_{r+2} \rightarrow i_{r+1} \text{ and } \mathbf{i} \in I_+^\gamma, \\ (\psi_{r+1}^\mathcal{O} \psi_r^\mathcal{O} \psi_{r+1}^\mathcal{O} + t^{1-\rho_r(\mathbf{i})}) f_{\mathbf{i}}^\mathcal{O}, & \text{if } i_r = i_{r+2} \leftarrow i_{r+1} \text{ and } \mathbf{i} \in I_-^\gamma, \\ (\psi_{r+1}^\mathcal{O} \psi_r^\mathcal{O} \psi_{r+1}^\mathcal{O} - 1) f_{\mathbf{i}}^\mathcal{O}, & \text{if } i_r = i_{r+2} \rightarrow i_{r+1} \text{ and } \mathbf{i} \in I_-^\gamma, \\ (\psi_{r+1}^\mathcal{O} \psi_r^\mathcal{O} \psi_{r+1}^\mathcal{O} + 1) f_{\mathbf{i}}^\mathcal{O}, & \text{if } i_r = i_{r+2} \leftarrow i_{r+1} \text{ and } \mathbf{i} \in I_+^\gamma, \\ \psi_{r+1}^\mathcal{O} \psi_r^\mathcal{O} \psi_{r+1}^\mathcal{O} f_{\mathbf{i}}^\mathcal{O}, & \text{otherwise,} \end{cases}$$

for all admissible  $\mathbf{i}, \mathbf{j} \in I^\gamma$  and all admissible  $r, s$  and  $t$  and where for  $d \in \mathbb{Z}$

$$\dot{y}_r^{\langle d \rangle} f_{\mathbf{i}}^\mathcal{O} = \begin{cases} (t^d \dot{y}_r^\mathcal{O} - [d]) f_{\mathbf{i}}^\mathcal{O}, & \text{if } \mathbf{i} \in I_+^\gamma, \\ (t^d \dot{y}_r^\mathcal{O} + [d]) f_{\mathbf{i}}^\mathcal{O}, & \text{if } \mathbf{i} \in I_-^\gamma. \end{cases}$$

If  $\mathbb{F}$  is an  $\mathcal{O}$ -module let  $\dot{R}_\gamma^\mathbb{F} = \dot{R}_\gamma^\mathcal{O} \otimes_{\mathcal{O}} \mathbb{F}$ .

To show that  $\dot{R}_\gamma^\mathcal{O}$  is finitely generated as an  $\mathcal{O}$ -module we need the following technical lemma, which is an analogue of [9, Lemma 4.31].

**2.44. Lemma.** *Suppose that  $1 \leq r \leq n$  and  $\mathbf{i} \in I^\gamma$ . If  $\mathbf{i} \notin I_+^\gamma \cup I_-^\gamma$  then  $f_{\mathbf{i}}^\mathcal{O} = 0$  and if  $\mathbf{i} \in I_\pm^\gamma$  then there exists a set  $X_r(\mathbf{i}) \subseteq e\mathbb{Z} \times \mathbb{N}$  such that*

$$\prod_{(c,m) \in X_r(\mathbf{i})} (\dot{y}_r^\mathcal{O} \mp [c])^m f_{\mathbf{i}}^\mathcal{O} = 0$$

in  $\dot{R}_\gamma^\mathcal{O}$ .

*Proof.* Arguing exactly as in proof of Lemma 1.16, if  $\mathbf{i} \in I^\gamma$  then  $f_{\mathbf{i}}^\mathcal{O} \neq 0$  only if  $i_1 = 0$  and  $i_2 = \pm 1$ . That is,  $f_{\mathbf{i}}^\mathcal{O} \neq 0$  only if  $\mathbf{i} \in I_+^\gamma \cup I_-^\gamma$ . Hence, we may assume that  $\mathbf{i} \in I_+^\gamma \cup I_-^\gamma$ .

Checking the relations in Definition 2.43,  $\dot{R}_\gamma^\mathcal{O}$  has an automorphism  $\#$  such that

$$(\psi_r^\mathcal{O})^\# = -\dot{\psi}_r^\mathcal{O}, \quad (\dot{y}_s^\mathcal{O})^\# = -\dot{y}_s^\mathcal{O} \quad \text{and} \quad (f_{\mathbf{i}}^\mathcal{O})^\# = f_{-\mathbf{i}}^\mathcal{O},$$

for all  $1 \leq r < n$ ,  $1 \leq s \leq n$  and  $\mathbf{i} \in I^\gamma$ . Therefore, it is enough to consider the case when  $\mathbf{i} \in I_+^\gamma$ .

By Definition 2.43,  $\dot{y}_1^\mathcal{O} f_{\mathbf{i}}^\mathcal{O} = 0$ , so we may take  $X_1(\mathbf{i}) = \{(0, 1)\}$ . As  $\psi_1^\mathcal{O} f_{\mathbf{i}}^\mathcal{O} = f_{s_1, \mathbf{i}}^\mathcal{O} \dot{\psi}_1^\mathcal{O}$ , it follows that  $\dot{\psi}_1^\mathcal{O} = 0$ . Therefore, if  $f_{\mathbf{i}}^\mathcal{O} \neq 0$  then  $0 = (\dot{\psi}_1^\mathcal{O})^2 f_{\mathbf{i}}^\mathcal{O} = (\dot{y}_1^\mathcal{O} - \dot{y}_2^\mathcal{O}) f_{\mathbf{i}}^\mathcal{O} = -\dot{y}_2^\mathcal{O} f_{\mathbf{i}}^\mathcal{O}$ , so  $\dot{y}_2^\mathcal{O} = 0$ . Hence, we can set  $X_2(\mathbf{i}) = \{(0, 1)\}$ , for all  $\mathbf{i} \in I^\gamma$ .

Now consider  $\dot{y}_3^\mathcal{O} f_{\mathbf{i}}^\mathcal{O}$ , for  $\mathbf{i} \in I_+^\gamma$ . If  $i_2 = i_3$  then the commutation relations for  $\dot{\psi}_2^\mathcal{O}$  and  $\dot{y}_3^\mathcal{O}$  give  $f_{\mathbf{i}}^\mathcal{O} = (\dot{y}_3^\mathcal{O} \dot{\psi}_2^\mathcal{O} - \dot{\psi}_2^\mathcal{O} \dot{y}_2^{\langle -e \rangle}) f_{\mathbf{i}}^\mathcal{O} = (\dot{y}_3^\mathcal{O} + [-e]) \dot{\psi}_2^\mathcal{O} f_{\mathbf{i}}^\mathcal{O}$  since  $\dot{y}_2^\mathcal{O} = 0$ . Similarly,  $f_{\mathbf{i}}^\mathcal{O} = \dot{\psi}_2^\mathcal{O} (\dot{y}_3^\mathcal{O} + [-e]) f_{\mathbf{i}}^\mathcal{O}$ . Therefore,  $f_{\mathbf{i}}^\mathcal{O} = (\dot{y}_3^\mathcal{O} + [e]) (\dot{\psi}_2^\mathcal{O})^2 (\dot{y}_3^\mathcal{O} + [-e]) f_{\mathbf{i}}^\mathcal{O} = 0$ . Hence, we can assume that  $i_2 \neq i_3$ . If  $i_2 \neq i_3$  and  $\mathbf{i} \in I_+^\gamma$  then

$$\begin{aligned} (\dot{y}_3^\mathcal{O} - [-e]) f_{\mathbf{i}}^\mathcal{O} &= (\dot{y}_3^\mathcal{O} - [-e]) (\dot{\psi}_2^\mathcal{O})^2 f_{\mathbf{i}}^\mathcal{O} = (\dot{y}_3^\mathcal{O} - [-e]) \dot{\psi}_2^\mathcal{O} f_{s_2, \mathbf{i}}^\mathcal{O} \dot{\psi}_2^\mathcal{O} \\ &= \dot{\psi}_2^\mathcal{O} (\dot{y}_2^\mathcal{O} + [-e] - [-e]) f_{s_2, \mathbf{i}}^\mathcal{O} \dot{\psi}_2^\mathcal{O} = 0. \end{aligned}$$



Hence, if  $i_2 \neq i_3$  set  $X_3(\mathbf{i}) = \{(-e, 1)\}$ . Similarly, if  $i_2 \rightarrow i_3$  then

$$(\dot{y}_3^\mathcal{O} - [-e])\dot{y}_3^\mathcal{O}\dot{f}_1^\mathcal{O} = (\dot{y}_3^\mathcal{O} - [-e])(\dot{y}_2^\mathcal{O} - \dot{\psi}_2^\mathcal{O})^2\dot{f}_1^\mathcal{O} = -(\dot{y}_3^\mathcal{O} - [-e])(\dot{\psi}_2^\mathcal{O})^2\dot{f}_1^\mathcal{O} = 0.$$

Consequently, we can set  $X_3(\mathbf{i}) = \{(-e, 1), (0, 1)\}$ . The case when  $i_2 \leftarrow i_3$  is similar and easier with  $X_3(\mathbf{i}) = \{(0, 1)\}$ .

The last two paragraphs show that if  $1 \leq r \leq 3$  and  $\mathbf{i} \in I_+^\gamma$  then there exists a set  $X_r(\mathbf{i}) \subseteq e\mathbb{Z} \times \mathbb{N}$  such that  $\prod_{(c,m) \in X_r(\mathbf{i})} (\dot{y}_r^\mathcal{O} - [c])^m \dot{f}_\mathbf{i}^\mathcal{O} = 0$ . If  $3 \leq r < n$  and  $\mathbf{i} \in I_+^\gamma$  then  $s_r \cdot \mathbf{i} \in I_+^\gamma$ . Moreover, the elements  $\dot{\psi}_3^\mathcal{O}, \dots, \dot{\psi}_{n-1}^\mathcal{O}$  and  $\dot{y}_4^\mathcal{O}, \dots, \dot{y}_n^\mathcal{O}$  satisfy the same defining relations as  $\psi_3^+, \dots, \psi_{n-1}^+, y_4^+, \dots, y_n^+$ . Therefore, the inductive argument in [9, Lemma 4.31] shows that there exists a set  $X_r(\mathbf{i}) \subseteq e\mathbb{Z} \times \mathbb{N}$  such that

$$\prod_{(c,m) \in X_r(\mathbf{i})} (\dot{y}_r^\mathcal{O} - [c])^m \dot{f}_\mathbf{i}^\mathcal{O} = 0, \quad \text{for } \mathbf{i} \in I_+^\gamma \text{ and } 1 \leq r \leq n.$$

(Note that if  $1 \leq r \leq 3$ , or if  $\mathbf{i} \in I_-^\gamma$ , then the argument from [9] does not apply because  $\dot{\psi}_1^\mathcal{O}$  and  $\dot{\psi}_2^\mathcal{O}$  satisfy slightly different relations to the corresponding elements considered in that paper.)  $\square$

Finally, we are able to prove the enhanced version of [Theorem 1.9](#) that we use to prove our main result. If  $A$  is an  $\mathcal{O}$ -algebra let  $\mathcal{H}_\xi^A(\mathfrak{S}_n) = \mathcal{H}_t^\mathcal{O} \otimes_\mathcal{O} A$ .

**2.45. Theorem.** *Suppose that  $\gamma \in Q_n^\varepsilon$  and that  $(\mathcal{O}, t)$  is the idempotent subring defined in (2.28). Then  $\dot{R}_\gamma^\mathcal{O} \cong \mathcal{H}_\gamma^\mathcal{O}$  as  $\mathcal{O}$ -algebras.*

*Proof.* By the results in [Section 2.4](#), from [Proposition 2.31](#) onwards, there is a unique surjective algebra homomorphism  $\dot{R}_\gamma^\mathcal{O} \twoheadrightarrow \mathcal{H}_\gamma^\mathcal{O}$  such that

$$\dot{\psi}_r^\mathcal{O} \mapsto \psi_r^\mathcal{O}, \quad \dot{y}_s^\mathcal{O} \mapsto y_s^\mathcal{O} \quad \text{and} \quad \dot{f}_\mathbf{i}^\mathcal{O} \mapsto f_\mathbf{i}^\mathcal{O},$$

for  $1 \leq r < n$ ,  $1 \leq s \leq n$  and  $\mathbf{i} \in I^\gamma$ . To prove that this map is an isomorphism we use the argument from [9, Theorem 4.32] to show that  $\dot{R}_\gamma^\mathcal{O}$  is free as an  $\mathcal{O}$ -module with the same rank as  $\mathcal{H}_\gamma^\mathcal{O}$ .

First, using the relations in [Definition 2.43](#) it is straightforward to show that  $\dot{R}_\gamma^\mathcal{O}$  is spanned by elements of the form  $f_w(\dot{y})\dot{\psi}_w^\mathcal{O}\dot{f}_\mathbf{i}^\mathcal{O}$ , where  $f_w(\dot{y})$  is a polynomial in  $\mathcal{O}[\dot{y}_1^\mathcal{O}, \dots, \dot{y}_n^\mathcal{O}]$ ,  $\mathbf{i} \in I^\gamma$  and for each  $w \in \mathfrak{S}_n$  we fix a reduced expression  $w = s_{r_1} \dots s_{r_k}$  and set  $\dot{\psi}_w^\mathcal{O} = \dot{\psi}_{r_1}^\mathcal{O} \dots \dot{\psi}_{r_k}^\mathcal{O}$ . Hence,  $\dot{R}_\gamma^\mathcal{O}$  is finitely generated as an  $\mathcal{O}$ -module by [Lemma 2.44](#).

Next, let  $\mathfrak{m} = x\mathcal{O}$  be the maximal ideal of  $\mathcal{O}$  and set  $\mathbb{F} = \mathcal{O}/\mathfrak{m}$  and  $\xi = t + \mathfrak{m} \in \mathbb{F}$ . Then  $\xi$  has quantum characteristic  $e$  because if  $k \in \mathbb{Z}$  then  $[k]_t \in \mathcal{J}(\mathcal{O}) = \mathfrak{m}$  if and only if  $k \in e\mathbb{Z}$  by [Definition 2.11](#). By [Definition 2.43](#), the relations in  $\dot{R}_\gamma^\mathbb{F}$  collapse and become the KLR relations for  $\mathcal{R}_e^\mathbb{F}(\mathfrak{S}_n)_\gamma$  given in [Definition 1.5](#). That is,  $\dot{R}_\gamma^\mathbb{F} \cong \mathcal{R}_e^\mathbb{F}(\mathfrak{S}_n)_\gamma$  as  $\mathbb{F}$ -algebras. Consequently,

$$\dim_{\mathbb{F}} \dot{R}_\gamma^\mathbb{F} = \dim_{\mathbb{F}} \mathcal{R}_e^\mathbb{F}(\mathfrak{S}_n)_\gamma = \text{rank}_{\mathcal{O}} \mathcal{H}_\gamma^\mathcal{O},$$

where the last equality follows by [3, Theorem 4.20] (alternatively, use [Theorem 1.9](#)). Since  $\mathfrak{m}$  is the unique maximal ideal of  $\mathcal{O}$ , and  $\dot{R}_\gamma^\mathcal{O}$  is finitely generated as an  $\mathcal{O}$ -module, Nakayama's Lemma implies that  $\dot{R}_\gamma^\mathcal{O}$  is free as an  $\mathcal{O}$ -module of rank  $\dim_{\mathbb{F}} \mathcal{H}_\gamma^\mathbb{F}(\mathfrak{S}_n)_\gamma = \text{rank}_{\mathcal{O}} \mathcal{H}_\gamma^\mathcal{O}$ . Hence, as an  $\mathcal{O}$ -module,  $\dot{R}_\gamma^\mathcal{O}$  is free of the same rank as  $\mathcal{H}_\gamma^\mathcal{O}$ . Since  $\mathcal{H}_\gamma^\mathcal{O}$  is also free over  $\mathcal{O}$ , it follows that the surjective algebra

homomorphism  $\dot{R}_\gamma^\mathcal{O} \rightarrow \mathcal{H}_\gamma^\mathcal{O}$  given in the first paragraph of the proof is actually an isomorphism and the theorem is proved.  $\square$

Recalling (2.17), for  $\gamma \in Q_n^\varepsilon$  define  $\mathcal{H}_\xi^\mathbb{F}(\mathfrak{A}_n)_\gamma = \mathcal{H}_t^\mathcal{O}(\mathfrak{A}_n)_\gamma \otimes_\mathcal{O} \mathbb{F}$ . Then  $\mathcal{H}_\xi^\mathbb{F}(\mathfrak{A}_n)_\gamma$  is a direct summand of  $\mathcal{H}_\xi^\mathbb{F}(\mathfrak{A}_n)$  by Corollary 2.18. By construction,  $F$  is (isomorphic to) a subfield of  $\mathbb{F}$  and the algebra  $\mathcal{H}_\xi^F(\mathfrak{A}_n)$  is the  $F$ -subalgebra of  $\mathcal{H}_\xi^\mathbb{F}(\mathfrak{A}_n)$  generated by the elements  $T_1, \dots, T_{n-1}$ . By Lemma 2.12 and Proposition 2.16,  $e_\gamma = f_\gamma^\mathcal{O} \otimes 1_\mathbb{F}$  is central idempotent in  $\mathcal{H}_\xi^\mathbb{F}(\mathfrak{A}_n)$ . Define

$$\mathcal{H}_\xi^F(\mathfrak{A}_n)_\gamma = \mathcal{H}_\xi^\mathbb{F}(\mathfrak{A}_n)_\gamma e_\gamma.$$

Then  $\mathcal{H}_\xi^F(\mathfrak{A}_n)_\gamma$  is the  $F$ -subalgebra of  $\mathcal{H}_\xi^\mathbb{F}(\mathfrak{A}_n)_\gamma$  generated by  $T_1 e_\gamma, \dots, T_{n-1} e_\gamma$ .

We are assuming that  $F$  is a field and that  $\xi \in F$  an element of quantum characteristic  $e$ . Recall from before Theorem A that a field  $F$  is **large enough** for  $\xi$  if  $F$  contains squareroots  $\sqrt{\xi}$  and  $\sqrt{1 + \xi + \xi^2}$  whenever  $e > 3$ . (In particular, if  $e = 3$  then any field is large enough for  $\xi$ .)

**2.46. Theorem.** *Suppose that  $\gamma \in Q_n^\varepsilon$ ,  $e > 2$  and that  $\xi \in F$  an element of quantum characteristic  $e$ . Let  $F$  be a large enough field for  $\xi$  of characteristic different from 2. Then  $\mathcal{H}_\xi^F(\mathfrak{A}_n)_\gamma \cong \mathcal{R}_e^F(\mathfrak{A}_n)_\gamma$ .*

*Proof.* Let  $(\mathcal{O}, t)$  be the idempotent subring given in Lemma 2.25, starting from  $F$  and  $\xi$ , and let  $\mathbb{F} = \mathcal{O}/\mathfrak{m}$ , where  $\mathfrak{m} = x\mathcal{O}$  is the maximal ideal of  $\mathcal{O}$ . Now  $\dot{R}_\gamma^\mathbb{F} \cong \mathcal{H}_\xi^\mathbb{F}(\mathfrak{S}_n)_\gamma$  by Theorem 2.45, so there is an isomorphism of  $\mathbb{F}$ -algebras  $\Theta: \mathcal{R}_e^\mathbb{F}(\mathfrak{S}_n)_\gamma \rightarrow \mathcal{H}_\xi^\mathbb{F}(\mathfrak{S}_n)_\gamma$  given by

$$\psi_r \otimes 1_\mathbb{F} \mapsto \psi_r^\mathcal{O} \otimes 1_\mathbb{F}, \quad y_s \otimes 1_\mathbb{F} \mapsto y_s^\mathcal{O} \otimes 1_\mathbb{F} \quad \text{and} \quad e(\mathbf{i}) \otimes 1_\mathbb{F} \mapsto f_{\mathbf{i}}^\mathcal{O} \otimes 1_\mathbb{F},$$

for  $1 \leq r < n$ ,  $1 \leq s \leq n$  and  $\mathbf{i} \in I^\gamma$ . By Corollary 2.30 the following diagram commutes:

$$\begin{array}{ccc} \mathcal{R}_e^\mathbb{F}(\mathfrak{S}_n)_\gamma & \xrightarrow{\Theta} & \mathcal{H}_\xi^\mathbb{F}(\mathfrak{S}_n)_\gamma \\ \text{sgn} \downarrow & & \downarrow \# \\ \mathcal{R}_e^\mathbb{F}(\mathfrak{S}_n)_\gamma & \xrightarrow{\Theta} & \mathcal{H}_\xi^\mathbb{F}(\mathfrak{S}_n)_\gamma \end{array}$$

Therefore,  $\Theta$  restricts to an isomorphism  $\Theta: \mathcal{R}_e^\mathbb{F}(\mathfrak{A}_n)_\gamma \rightarrow \mathcal{H}_\xi^\mathbb{F}(\mathfrak{A}_n)_\gamma$ .

We have now shown that  $\mathcal{R}_e^\mathbb{F}(\mathfrak{A}_n)_\gamma$  and  $\mathcal{H}_\xi^\mathbb{F}(\mathfrak{A}_n)_\gamma$  are isomorphic over  $\mathbb{F}$  but, of course, we want the isomorphism over  $F$ , which is a subfield of  $\mathbb{F}$ . Since  $F$  is large enough for  $\xi$ , by Definition 2.29 the generators of  $\mathcal{H}_\xi^\mathbb{F}(\mathfrak{S}_n)_\gamma$  listed in Proposition 2.31 all belong to  $\mathcal{H}_\xi^F(\mathfrak{S}_n)_\gamma$ , which we consider as a subalgebra of  $\mathcal{R}_e^\mathbb{F}(\mathfrak{S}_n)_\gamma$ . The coefficients in the relations of Definition 2.43 also belong to  $\mathcal{H}_\xi^F(\mathfrak{S}_n)_\gamma$ . Hence, there is a surjective algebra homomorphism  $\mathcal{R}_e^F(\mathfrak{S}_n)_\gamma \rightarrow \mathcal{H}_\xi^F(\mathfrak{S}_n)_\gamma$ . Counting dimensions, this map is an isomorphism so  $\mathcal{R}_e^F(\mathfrak{S}_n)_\gamma \cong \mathcal{H}_\xi^F(\mathfrak{S}_n)_\gamma$  as  $F$ -algebras. Applying Corollary 2.30, as above, it follows that  $\mathcal{R}_e^F(\mathfrak{A}_n)_\gamma \cong \mathcal{H}_\xi^F(\mathfrak{A}_n)_\gamma$  as required.  $\square$

In view of Corollary 2.18, we obtain Theorem A from the introduction.

**2.47. Corollary.** *Let  $F$  be a field of characteristic different from 2 and let  $\xi \in F$  be an element of quantum characteristic  $e$ . Suppose that  $e > 2$  and that  $F$  is a large enough field for  $\xi$ . Then  $\mathcal{H}_\xi^F(\mathfrak{A}_n) \cong \mathcal{R}_e^F(\mathfrak{A}_n)$ .*

Hence, as noted in [Corollary A1](#), the alternating Hecke algebra  $\mathcal{H}_\xi^F(\mathfrak{A}_n)$  is a  $\mathbb{Z}$ -graded algebra. In particular,  $F\mathfrak{A}_n$  is a  $\mathbb{Z}$ -graded algebra when  $F$  is large enough for  $\xi = 1$ .

### 3. A HOMOGENEOUS BASIS FOR $\mathcal{H}_\xi^F(\mathfrak{A}_n)$

We have now proved [Theorem A](#) and [Theorem B](#) from the introduction. It remains to prove [Theorem C](#), which gives the graded dimension of  $\mathcal{R}_e(\mathfrak{A}_n)$ . To do this we give a homogeneous basis for  $\mathcal{R}_e(\mathfrak{A}_n)$  by combining the two graded cellular bases of  $\mathcal{R}_e(\mathfrak{S}_n)$  defined by Hu and the second-named author [\[7\]](#). In order to define these bases we need some definitions.

Fix a partition  $\lambda \in \mathcal{P}_n$ . If  $A = (r, c)$  and  $B = (s, d)$  are nodes of  $\lambda$  then  $A$  is **strictly above**  $B$ , or  $B$  is **strictly below**  $A$ , if  $r < s$ . Following Brundan, Kleshchev and Wang [\[4, §1\]](#), define integers

$$\begin{aligned} d_A(\lambda) &= \# \left\{ \begin{array}{c} \text{addable } i\text{-nodes of } \lambda \\ \text{strictly below } A \end{array} \right\} - \# \left\{ \begin{array}{c} \text{removable } i\text{-nodes of } \lambda \\ \text{strictly below } A \end{array} \right\}, \\ d^A(\lambda) &= \# \left\{ \begin{array}{c} \text{addable } i\text{-nodes of } \lambda \\ \text{strictly above } A \end{array} \right\} - \# \left\{ \begin{array}{c} \text{removable } i\text{-nodes of } \lambda \\ \text{strictly above } A \end{array} \right\}. \end{aligned}$$

**3.1. Definition** (Brundan, Kleshchev and Wang [\[4, §1\]](#)). Let  $\mathbf{t}$  be a standard  $\lambda$ -tableau, for  $\lambda \in \mathcal{P}_n$ , and let  $A = \mathbf{t}^{-1}(n)$ . Then the **degree**  $\deg \mathbf{t}$  and **codeg**  $\text{codeg } \mathbf{t}$  of  $\mathbf{t}$  are defined inductively by

$$\begin{aligned} \deg \mathbf{t} &= \begin{cases} \deg \mathbf{t}_{\downarrow(n-1)} + d_A(\lambda), & \text{if } n > 0, \\ 1, & \text{if } n = 0, \end{cases} \\ \text{codeg } \mathbf{t} &= \begin{cases} \text{codeg } \mathbf{t}_{\downarrow(n-1)} + d^A(\lambda), & \text{if } n > 0, \\ 1, & \text{if } n = 0, \end{cases} \end{aligned}$$

Fix  $\lambda \in \mathcal{P}_n$ . If  $1 \leq m \leq n$  and  $\mathbf{t} \in \text{Std}(\lambda)$  let  $\text{col}_m(\mathbf{t}) = c$  if  $m$  appears in column  $c$  of  $\mathbf{t}$  and let  $\text{row}_m(\mathbf{t}) = r$  if  $m$  appears in row  $r$  of  $\mathbf{t}$ . Following [\[7\]](#) set

$$y_\lambda = \prod_{\substack{1 \leq m \leq n \\ \text{col}_m(\mathbf{t}^\lambda) \equiv 0 \pmod{e}}} y_m \quad \text{and} \quad y'_\lambda = \prod_{\substack{1 \leq m \leq n \\ \text{row}_m(\mathbf{t}_\lambda) \equiv 0 \pmod{e}}} y_m.$$

As we are considering the special case when  $\Lambda = \Lambda_0$  the definitions of  $y_\lambda$  and  $y'_\lambda$  from [\[7\]](#) simplify and are equivalent to the formulas above.

If  $\mathbf{t} \in \text{Std}(\lambda)$  define permutations  $d(\mathbf{t})$  and  $d'(\mathbf{t})$  in  $\mathfrak{S}_n$  by

$$\mathbf{t} = d(\mathbf{t})\mathbf{t}^\lambda \quad \text{and} \quad \mathbf{t} = d'(\mathbf{t})\mathbf{t}_\lambda,$$

where  $\mathfrak{S}_n$  acts on  $\mathbf{t}$  by permuting its entries. For each  $w \in \mathfrak{S}_n$  fix a reduced expression  $w = s_{r_1} \dots s_{r_k}$  and define  $\psi_w = \psi_{r_1} \dots \psi_{r_k} \in \mathcal{R}_e(\mathfrak{S}_n)$ . In general,  $\psi_w$  depends upon the choice of reduced expression for  $w$ .

**3.2. Definition** (Hu and Mathas [\[7, Definitions 5.1 and 6.9\]](#)). Suppose that  $\mathbf{s}, \mathbf{t} \in \text{Std}(\lambda)$ , for  $\lambda \in \mathcal{P}_n$ . Set  $\mathbf{i}^\lambda = \text{res}(\mathbf{t}^\lambda)$  and  $\mathbf{i}_\lambda = \text{res}(\mathbf{t}_\lambda)$  and define

$$\psi_{\mathbf{st}} = \psi_{d(\mathbf{s})} y_\lambda e(\mathbf{i}^\lambda) \psi_{d(\mathbf{t})}^* \quad \text{and} \quad \psi'_{\mathbf{st}} = \psi_{d'(\mathbf{s})} y'_\lambda e(\mathbf{i}_\lambda) \psi_{d'(\mathbf{t})}^*.$$

By construction,  $\psi_{\mathbf{st}}$  and  $\psi'_{\mathbf{st}}$  are homogeneous elements of  $\mathcal{R}_e(\mathfrak{S}_n)$ .

3.3. *Remark.* For the reasons explained in [8, Remark 3.12], we are following the conventions of [8, 15] here rather than those of [7]. In particular, the element  $\psi'_{st}$  defined above is equal to  $\psi'_{s't'}$  in the notation of [7].

3.4. **Theorem** (Hu and Mathas [7] and Li [16]). *Let  $\mathcal{Z}$  be a commutative ring and suppose that  $e > 2$  and  $n \geq 0$ . Then  $\mathcal{R}_e(\mathfrak{S}_n)$  is free as a  $\mathcal{Z}$ -module with homogeneous bases  $\{\psi_{st} \mid s, t \in \text{Std}^2(\mathcal{P}_n)\}$  and  $\{\psi'_{st} \mid s, t \in \text{Std}^2(\mathcal{P}_n)\}$ . Moreover, if  $(s, t) \in \text{Std}^2(\mathcal{P}_n)$  then  $\deg \psi_{st} = \deg s + \deg t$  and  $\deg \psi'_{st} = \text{codeg } s + \text{codeg } t$ .*

This result was first proved in [7] with some restrictions on the ring  $\mathcal{Z}$ . Li's [16] extension of this result to arbitrary rings is a difficult theorem. A second proof of this result, using geometry, is given in [23].

Although we will not need this, by [7, Theorems 5.8 and 6.11] both of the bases of Theorem 3.4 are graded *cellular* bases of  $\mathcal{R}_e(\mathfrak{S}_n)$  in the sense of Graham and Lehrer [6, 7].

3.5. **Example.** Set  $n = e = 3$  and set

$$s = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \quad t = \begin{bmatrix} 1 & 2 \\ 3 \end{bmatrix}, \quad u = \begin{bmatrix} 1 & 3 \\ 2 \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Using the definitions,

$$\begin{aligned} \psi_{ss} &= y_3 e(012), & \psi'_{vv} &= y_3 e(021), \\ \psi_{tt} &= e(012), & \psi'_{uu} &= e(021), \\ \psi_{tu} &= e(012)\psi_2, & \psi'_{ut} &= e(021)\psi_2, \\ \psi_{ut} &= \psi_2 e(012), & \psi'_{tu} &= \psi_2 e(021), \\ \psi_{uu} &= \psi_2 e(012)\psi_2, & \psi'_{tt} &= \psi_2 e(021)\psi_2, \\ \psi_{vv} &= e(021), & \psi'_{ss} &= e(012). \end{aligned}$$

Notice that  $\psi_{tu} = \psi_2 e(021) = \psi'_{tu}$ ,  $\psi_{uu} = \psi_2^2 e(021) = -y_3 e(021) = -\psi'_{vv}$  and, similarly,  $\psi'_{tt} = -y_3 e(012) = -\psi_{ss}$ . Therefore, up to sign, the  $\psi$  and  $\psi'$  bases coincide with the basis of  $\mathcal{R}_e(\mathfrak{S}_3)$  given in Example 1.8.  $\diamond$

We now use the two homogeneous bases for  $\mathcal{R}_e(\mathfrak{S}_n)$  from Theorem 3.4 to construct homogeneous bases for  $\mathcal{R}_e(\mathfrak{A}_n)$ . The next result, which follows easily from the definitions, shows that the  $\psi$  and  $\psi'$ -bases are interchanged by the sign automorphism.

3.6. **Lemma** (Hu and Mathas [8, Proposition 3.26]). *Suppose that  $s, t \in \text{Std}(\mathcal{P}_n)$ . Then  $\psi_{st}^{\text{sgn}} = (-1)^{\ell(d(s)) + \ell(d(t)) + \deg t^\lambda} \psi'_{s't'}$ . In particular,  $\deg \psi_{st} = \deg \psi'_{s't'}$ .*

In fact, it follows easily from the definitions that  $\deg t = \text{codeg } t'$  for any standard tableau  $t$ . Hence,  $\deg \psi'_{s't'} = \text{codeg } s' + \text{codeg } t' = \deg s + \deg t = \deg \psi_{st}$  as claimed.

Recall from Proposition 1.19 that  $\mathcal{R}_e(\mathfrak{S}_n)_\gamma \cong \mathcal{R}_\gamma^\varepsilon = \mathcal{R}_\gamma^{\varepsilon+} \oplus \mathcal{R}_\gamma^{\varepsilon-}$  is a  $(\mathbb{Z}_2 \times \mathbb{Z})$ -graded algebra and that  $\mathcal{R}_e(\mathfrak{A}_n)_\gamma \cong \mathcal{R}_\gamma^{\varepsilon+}$  by Corollary 1.21. To find a basis for  $\mathcal{R}_e(\mathfrak{A}_n)_\gamma$  we first give a basis of  $\mathcal{R}_e(\mathfrak{S}_n)_\gamma$  that is homogeneous with respect to the  $(\mathbb{Z}_2 \times \mathbb{Z})$ -grading.

3.7. **Definition.** Fix  $e > 2$  and  $\lambda \in \mathcal{P}_n$ . For  $s, t \in \text{Std}(\lambda)$  define elements

$$\Psi_{st}^+ = \psi_{st} + \psi_{st}^{\text{sgn}} \quad \text{and} \quad \Psi_{st}^- = \psi_{st} - \psi_{st}^{\text{sgn}}.$$

Fix  $(s, t) \in \text{Std}^2(\mathcal{P}_n)$ . By [Lemma 3.6](#),  $\Psi_{st}^+$  and  $\Psi_{st}^-$  are homogeneous with respect to the  $\mathbb{Z}$ -grading on  $\mathcal{R}_e(\mathfrak{S}_n)$ . Furthermore, by [Corollary 1.21](#),  $\Psi_{st}^+$  is even and  $\Psi_{st}^-$  is odd with respect to the  $\mathbb{Z}_2$ -grading. Hence, the elements  $\Psi_{st}^\pm$  are homogeneous with respect to the  $(\mathbb{Z}_2 \times \mathbb{Z})$ -grading on  $\mathcal{R}_e(\mathfrak{S}_n)$ .

Fix  $(s, t) \in \text{Std}^2(\mathcal{P}_n)$  and set  $\mathbf{i}^s = \text{res}(s)$  and  $\mathbf{i}^t = \text{res}(t)$ . By [\[8, \(3.13\)\]](#), if  $\mathbf{i}, \mathbf{j} \in I^n$

$$(3.8) \quad e(\mathbf{i})\psi_{st}e(\mathbf{j}) = \delta_{\mathbf{i}\mathbf{i}}\delta_{\mathbf{i}\mathbf{j}}\psi_{st} \quad \text{and} \quad e(\mathbf{i})\psi'_{st}e(\mathbf{j}) = \delta_{\mathbf{i}\mathbf{i}}\delta_{\mathbf{i}\mathbf{j}}\psi'_{st}$$

Set  $\varepsilon_1(\mathbf{i}) = e(\mathbf{i}) - e(-\mathbf{i})$  and  $\varepsilon_0(\mathbf{i}) = \varepsilon_1(\mathbf{i})^2 = e(\mathbf{i}) + e(-\mathbf{i})$ . Observe that [\(3.8\)](#) implies that if  $\mathbf{i} \in I_+^\gamma$  and  $(s, t) \in \text{Std}(\mathcal{P}_n)$  with  $\mathbf{i}^s = \text{res}(s) \in I^\gamma$  and  $\mathbf{i}^t = \text{res}(t) \in I^\gamma$  then

$$(3.9) \quad \varepsilon_1(\mathbf{i})\Psi_{st}^+ \begin{cases} \Psi_{st}^- & \text{if } \mathbf{i} = \mathbf{i}^s, \\ -\Psi_{st}^- & \text{if } \mathbf{i} = -\mathbf{i}^s, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad \Psi_{st}^+\varepsilon_1(\mathbf{i}) = \begin{cases} \Psi_{st}^- & \text{if } \mathbf{i} = \mathbf{i}^t, \\ -\Psi_{st}^- & \text{if } \mathbf{i} = -\mathbf{i}^t, \\ 0, & \text{otherwise.} \end{cases}$$

Since  $\varepsilon_1(-\mathbf{i}) = -\varepsilon_1(\mathbf{i})$  and  $\varepsilon_0(\mathbf{i}) = \varepsilon_1(\mathbf{i})^2$ , this readily implies the corresponding formulas for the action of  $\varepsilon_a(\mathbf{i})$  on  $\Psi_{st}^\pm$ , for all  $a \in \mathbb{Z}_2$  and  $\mathbf{i} \in I^\gamma$ .

The **dominance** order on  $\mathcal{P}_n$  is the partial order  $\trianglerighteq$  given by  $\lambda \trianglerighteq \mu$  if

$$\sum_{j=1}^k \lambda_j \geq \sum_{j=1}^k \mu_j \quad \text{for all } k \geq 0$$

Write  $\lambda \triangleright \mu$  if  $\lambda \trianglerighteq \mu$  and  $\lambda \neq \mu$ .

For  $\lambda \in \mathcal{P}_n$  define  $\text{Std}_+(\lambda) = \{s \in \text{Std}(\lambda) \mid \text{res}(s) \in I_+^n\}$ . We will use this set to index a homogeneous basis for  $\mathcal{R}_e(\mathfrak{S}_n)$ , with respect to its  $(\mathbb{Z}_2 \times \mathbb{Z})$ -grading. The following simple combinatorial result is probably well-known.

**3.10. Lemma.** *Suppose that  $n \geq 0$ . Then*

$$\sum_{\substack{\lambda \in \mathcal{P}_n \\ \lambda \triangleright \lambda'}} |\text{Std}_+(\lambda)| \cdot |\text{Std}(\lambda)| = \frac{n!}{2}.$$

*Proof.* Implicit in [Theorem 3.4](#), is the well-known fact that  $n! = \sum_{\lambda \in \mathcal{P}_n} |\text{Std}(\lambda)|^2$ . Since  $|\text{Std}(\lambda)| = |\text{Std}(\lambda')|$ , via the map  $\mathbf{t} \mapsto \mathbf{t}'$ , it follows that

$$\begin{aligned} \frac{n!}{2} &= \sum_{\substack{\lambda \in \mathcal{P}_n \\ \lambda \triangleright \lambda'}} |\text{Std}(\lambda)|^2 + \frac{1}{2} \sum_{\substack{\lambda \in \mathcal{P}_n \\ \lambda = \lambda'}} |\text{Std}(\lambda)|^2 \\ &= \sum_{\substack{\lambda \in \mathcal{P}_n \\ \lambda \triangleright \lambda'}} (|\text{Std}_+(\lambda)| + |\text{Std}_+(\lambda')|) \cdot |\text{Std}(\lambda)| + \sum_{\substack{\lambda \in \mathcal{P}_n \\ \lambda = \lambda'}} |\text{Std}_+(\lambda)| \cdot |\text{Std}(\lambda)|. \end{aligned}$$

□

Recall from [Section 1.5](#) that  $\text{Deg} : \mathcal{R}_e(\mathfrak{S}_n) \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}$  is the degree function for the  $(\mathbb{Z}_2 \times \mathbb{Z})$ -grading on  $\mathcal{R}_e(\mathfrak{S}_n)$ .

**3.11. Theorem.** *Let  $\mathcal{Z}$  be a commutative ring such that 2 is invertible in  $\mathcal{Z}$  and suppose that  $e > 2$  and  $n \geq 0$ . Then  $\mathcal{R}_e(\mathfrak{S}_n)$  is free as a  $\mathcal{Z}$ -module with basis*

$$\{\Psi_{st}^+, \Psi_{st}^- \mid s \in \text{Std}_+(\lambda), t \in \text{Std}(\lambda) \text{ for } \lambda \in \mathcal{P}_n\}.$$

*Moreover, this basis is homogeneous with respect to the  $(\mathbb{Z}_2 \times \mathbb{Z})$ -grading on  $\mathcal{R}_e(\mathfrak{S}_n)$ .*

*Proof.* We have already noted  $\Psi_{\mathbf{st}}^\pm$  is homogeneous with respect to the  $(\mathbb{Z}_2 \times \mathbb{Z})$ -grading. More precisely, if  $(\mathbf{s}, \mathbf{t}) \in \text{Std}^2(\mathcal{P}_n)$  then

$$\text{Deg } \Psi_{\mathbf{st}}^+ = (0, \deg \mathbf{s} + \deg \mathbf{t}) \quad \text{and} \quad \text{Deg } \Psi_{\mathbf{st}}^- = (1, \deg \mathbf{s} + \deg \mathbf{t}).$$

Let  $R_n$  be the  $\mathcal{Z}$ -submodule of  $\mathcal{R}_e(\mathfrak{S}_n)$  spanned by the elements in the statement of the theorem. Fix  $\mathbf{s} \in \text{Std}_+(\lambda)$  and  $\mathbf{t} \in \text{Std}(\lambda)$ , for some  $\lambda \in \mathcal{P}_n$ . Since 2 is invertible in  $\mathcal{Z}$ ,

$$\psi_{\mathbf{st}} = \frac{1}{2}(\Psi_{\mathbf{st}}^+ + \Psi_{\mathbf{st}}^-) \quad \text{and} \quad \psi_{\mathbf{st}}^{\text{sgn}} = \frac{1}{2}(\Psi_{\mathbf{st}}^+ - \Psi_{\mathbf{st}}^-).$$

Hence,  $\psi_{\mathbf{st}}, \psi'_{\mathbf{s}'\mathbf{t}'} \in R_n$  since  $\psi'_{\mathbf{s}'\mathbf{t}'} = \pm \psi_{\mathbf{st}}^{\text{sgn}}$  by Lemma 3.6. Recall from Lemma 1.16 that  $e(\mathbf{i}) \neq 0$  only if  $\mathbf{i} \in I_+^n$  or  $\mathbf{i} \in I_-^n$ . Set  $e_+ = \sum_{\mathbf{i} \in I_+^n} e(\mathbf{i})$  and  $e_- = \sum_{\mathbf{i} \in I_-^n} e(\mathbf{i})$ . By Theorem 3.4 and (3.8), as  $\mathcal{Z}$ -modules,

$$e_+ \mathcal{R}_e(\mathfrak{S}_n) = \langle \psi_{\mathbf{st}} \mid (\mathbf{s}, \mathbf{t}) \in \text{Std}^2(\mathcal{P}_n) \text{ and } \text{res}(\mathbf{s}) \in I_+^n \rangle_{\mathcal{Z}} \subseteq e_+ R_n$$

Similarly, since  $\text{res}(\mathbf{s}') = -\text{res}(\mathbf{s})$ , Theorem 3.4 also implies that

$$e_- \mathcal{R}_e(\mathfrak{S}_n) = \langle \psi'_{\mathbf{st}} \mid (\mathbf{s}, \mathbf{t}) \in \text{Std}^2(\mathcal{P}_n) \text{ and } \text{res}(\mathbf{s}) \in I_-^n \rangle_{\mathcal{Z}} \subseteq e_- R_n.$$

Hence,  $e_+ R_n \subseteq e_+ \mathcal{R}_e(\mathfrak{S}_n) \subseteq e_+ R_n$  and  $e_- R_n \subseteq e_- \mathcal{R}_e(\mathfrak{S}_n) \subseteq e_- R_n$ , so that

$$\mathcal{R}_e(\mathfrak{S}_n) = e_+ \mathcal{R}_e(\mathfrak{S}_n) \oplus e_- \mathcal{R}_e(\mathfrak{S}_n) = e_+ R_n \oplus e_- R_n = R_n.$$

We have now shown that the set of elements  $\{\Psi_{\mathbf{st}}^\pm\}$  in the statement of the theorem span  $\mathcal{R}_e(\mathfrak{S}_n)$ . Let  $F$  be the field of fractions of  $\mathcal{Z}$ . Using Lemma 3.10 to count dimensions, it follows that  $\{\Psi_{\mathbf{st}}^\pm \otimes 1_F\}$  is a basis of  $\mathcal{R}_e^F(\mathfrak{S}_n) \cong \mathcal{R}_e(\mathfrak{S}_n) \otimes_{\mathcal{Z}} F$ . Hence,  $\{\Psi_{\mathbf{st}}^\pm\}$  is  $\mathcal{Z}$ -linearly independent, completing the proof.  $\square$

By Corollary 1.21,  $\mathcal{R}_e(\mathfrak{A}_n)$  is the even component of  $\mathcal{R}_e(\mathfrak{S}_n)$ , with respect to the  $\mathbb{Z}_2$ -grading. Hence, we have the following.

**3.12. Corollary.** *Let  $\mathcal{Z}$  be a commutative ring such that 2 is invertible in  $\mathcal{Z}$  and suppose that  $e > 2$  and  $n \geq 0$ . Then  $\mathcal{R}_e(\mathfrak{A}_n)$  is free as a  $\mathcal{Z}$ -module with basis*

$$\{\Psi_{\mathbf{st}}^+ \mid \mathbf{s} \in \text{Std}_+(\lambda) \text{ and } \mathbf{t} \in \text{Std}(\lambda) \text{ for } \lambda \in \mathcal{P}_n\}.$$

Moreover, this basis is homogeneous with respect to the  $\mathbb{Z}$ -grading on  $\mathcal{R}_e(\mathfrak{A}_n)$ .

**3.13. Example.** Continuing the notation of Example 3.5,

$$\Psi_{\mathbf{ss}}^+ = \mathcal{V}_3 = -\Psi_{\mathbf{uu}}^+ \quad \Psi_{\mathbf{tt}}^+ = 1 = \Psi_{\mathbf{vv}}^+ \quad \text{and} \quad \Psi_{\mathbf{tu}}^+ = \Psi_2^+ = -\Psi_{\mathbf{tu}}^+.$$

Hence,  $\{\Psi_{\mathbf{ab}}^+ \mid \mathbf{a} \in \text{Std}_+(\lambda) \text{ and } \mathbf{b} \in \text{Std}(\lambda)\} = \{\Psi_{\mathbf{ss}}^+, \Psi_{\mathbf{tt}}^+, \Psi_{\mathbf{tu}}^+\}$  is the basis of  $\mathcal{R}_e(\mathfrak{A}_n)$  constructed in Example 1.13.  $\diamond$

If  $M = \bigoplus_{d \in \mathbb{Z}} M_d$  is a  $\mathbb{Z}$ -graded module then its **graded dimension** is the Laurent polynomial

$$\dim_q M = \sum_{d \in \mathbb{Z}} (\dim M_d) q^d \in \mathcal{A} = \mathbb{N}[q, q^{-1}],$$

where  $q$  is an indeterminate over  $\mathbb{Z}$ . Adding up the degrees of the homogeneous basis elements in Corollary 1.21 gives Theorem C from the introduction.

**3.14. Corollary.** *Let  $F$  be a field of characteristic different from 2 and suppose that  $e > 2$  and  $n \geq 0$ . Then the graded dimension of  $\mathcal{R}_e(\mathfrak{A}_n)$  is*

$$\dim_q \mathcal{R}_e(\mathfrak{A}_n) = \sum_{\lambda \in \mathcal{P}_n} \sum_{\substack{\mathbf{s} \in \text{Std}_+(\lambda) \\ \mathbf{t} \in \text{Std}(\lambda)}} q^{\deg \mathbf{s} + \deg \mathbf{t}}.$$

By (3.9), if  $\gamma \in Q_n^\varepsilon$  then the basis of  $\mathcal{R}_e(\mathfrak{A}_n)$  given in Corollary 3.12 restricts to give a basis of  $\mathcal{R}_e(\mathfrak{A}_n)_\gamma$ . Note that if  $(\mathbf{s}, \mathbf{t}) \in \text{Std}^2(\mathcal{P}_n)$  and  $\text{res}(\mathbf{s}) \in I^\gamma$  then  $\text{res}(\mathbf{t}) \in I^\gamma$ .

**3.15. Corollary.** *Fix  $\gamma \in Q_n^\varepsilon$  and let  $\mathcal{Z}$  be a commutative ring such that 2 is invertible in  $\mathcal{Z}$  and suppose that  $e > 2$  and  $n \geq 0$ . Then  $\mathcal{R}_e(\mathfrak{A}_n)_\gamma$  is free as a  $\mathcal{Z}$ -module with basis  $\{\Psi_{\mathbf{st}}^+ \mid (\mathbf{s}, \mathbf{t}) \in \text{Std}^2(\mathcal{P}_n) \text{ for } \text{res}(\mathbf{s}) \in I_+^\gamma\}$ .*

We next show that  $\mathcal{R}_e(\mathfrak{A}_n)_\gamma$  is a graded symmetric algebra. Repeating the arguments leading to Corollary 3.15 we obtain a second homogeneous basis for  $\mathcal{R}_e(\mathfrak{A}_n)_\gamma$ . We will use the two homogeneous bases of  $\mathcal{R}_e(\mathfrak{A}_n)_\gamma$  to prove that  $\mathcal{R}_e(\mathfrak{A}_n)_\gamma$  is graded symmetric.

**3.16. Corollary.** *Fix  $\gamma \in Q_n^\varepsilon$  and let  $\mathcal{Z}$  be a commutative ring such that 2 is invertible in  $\mathcal{Z}$  and suppose that  $e > 2$  and  $n \geq 0$ . Then  $\mathcal{R}_e(\mathfrak{A}_n)_\gamma$  is free as a  $\mathcal{Z}$ -module with basis  $\{\Psi_{\mathbf{st}}^+ \mid (\mathbf{s}, \mathbf{t}) \in \text{Std}^2(\mathcal{P}_n) \text{ for } \text{res}(\mathbf{s}) \in I_-^\gamma\}$ .*

Before we show that the blocks of  $\mathcal{R}_e(\mathfrak{A}_n)$  are graded symmetric algebras we recall some definitions. A **trace form** on an algebra  $A$  is a linear map  $\tau : A \rightarrow F$  such that  $\tau(ab) = \tau(ba)$ , for all  $a, b \in A$ . The algebra  $A$  is **symmetric** if  $A$  is equipped with a non-degenerate symmetric bilinear form  $\theta : A \times A \rightarrow F$  which is associative in the following sense:

$$\theta(xy, z) = \theta(x, yz), \quad \text{for all } x, y, z \in A.$$

A graded algebra  $A$  is a **graded symmetric** algebra if there exists a non-degenerate homogeneous trace form  $\tau : A \rightarrow F$ . Suppose that  $A$  is equipped with a homogeneous anti-isomorphism  $\sigma$  of order 2. In view of [7, Lemma 6.13], giving a non-degenerate trace form  $\tau$  is equivalent to requiring that the bilinear form

$$\langle a, b \rangle = \tau(ab^\sigma) \quad \text{for all } a, b \in A$$

is non-degenerate. We work interchangeably with the trace form  $\tau$  and its associated bilinear form.

The algebras  $\mathcal{R}_e(\mathfrak{S}_n)$  and  $\mathcal{R}_e(\mathfrak{A}_n)$  are both symmetric algebras but their homogeneous trace forms are defined on the blocks  $\mathcal{R}_e(\mathfrak{A}_n)_\gamma$  of these algebras, for  $\gamma \in Q_n^\varepsilon$ . Recall that  $Q_n^\varepsilon = Q_n^+ / \sim$ , where  $\alpha \sim \alpha'$  if  $(\Lambda_i, \alpha) = (\Lambda_{-i}, \alpha')$ , for all  $i \in I$ . Fix  $\alpha \in Q_n^+$ . The **defect** of  $\alpha$  is the non-negative integer

$$\text{def } \alpha = (\Lambda_0, \alpha) - \frac{1}{2}(\alpha, \alpha),$$

where  $(\ , \ ) : P^+ \times Q^+ \rightarrow \mathbb{Z}$  is the pairing defined in Section 1.3. Hence, if  $\alpha \sim \alpha'$  then  $\text{def } \alpha = \text{def } \alpha'$ . Therefore, if  $\gamma \in Q_n^\varepsilon$  we can define the **defect** of  $\gamma$  to be  $\text{def } \gamma = \text{def } \alpha$ , for any  $\alpha \in \gamma$ . Set

$$\mathcal{P}_\alpha = \{\lambda \in \mathcal{P}_n \mid \mathbf{i}^\lambda \in I^\alpha\} \quad \text{and} \quad \mathcal{P}_\gamma = \bigcup_{\alpha \in \gamma} \mathcal{P}_\alpha,$$



for  $\alpha \in Q_n^+$  and  $\gamma \in Q_n^\varepsilon$ . In the usual language from the representation theory of the symmetric groups, the partitions in  $\mathcal{P}_\gamma$  have ***e-weight***  $\text{def } \gamma$ .

The following useful fact is straightforward to establish from the definitions.

**3.17. Lemma** (Brundan and Kleshchev [4, Lemma 3.11, 3.12]). *Let  $\alpha \in Q_n^+$  and suppose that  $\mathbf{s} \in \text{Std}(\mathcal{P}_\alpha)$ . Then  $\deg \mathbf{s} + \text{codeg } \mathbf{s} = \text{def } \alpha$ .*

As is well-known and easy to prove (see, for example, [17, Proposition 1.16]), the Iwahori-Hecke  $\mathcal{H}_\xi(\mathfrak{S}_n)$  is a symmetric algebra with trace form  $\tau$ . Explicitly, if  $h = \sum_{w \in \mathfrak{S}_n} a_w T_w \in \mathcal{H}_\xi(\mathfrak{S}_n)$  then  $\tau(h) = a_1$ . For  $\alpha \in Q^+$  let  $\tau_\alpha$  be the homogeneous component of  $\tau$  of degree  $-2 \text{def } \alpha$  restricted to  $\mathcal{H}_\xi(\mathfrak{S}_n)_\alpha$ . Let  $\langle \cdot, \cdot \rangle_\alpha$  be the homogeneous bilinear form associated with  $\tau$ .

Let  $\star$  be the unique homogeneous anti-isomorphism of  $\mathcal{R}_e(\mathfrak{S}_n)$  that fixes each of the generators of  $\mathcal{R}_e(\mathfrak{S}_n)$ . Then  $\psi_{\mathbf{st}}^\star = \psi_{\mathbf{ts}}$  and  $(\psi_{\mathbf{st}}')^\star = \psi_{\mathbf{ts}}'$ , for all  $(\mathbf{s}, \mathbf{t}) \in \text{Std}^2(\mathcal{P}_n)$ .

The following result was first proved in [7].

**3.18. Theorem** (Hu and Mathas [7, Theorem 6.7]). *Suppose that  $\alpha \in Q_n^+$  and that  $F$  is a field. Then the KLR algebra  $\mathcal{R}_e(\mathfrak{S}_n)_\alpha$  is a graded symmetric algebra with homogeneous bilinear form  $\langle \cdot, \cdot \rangle_\alpha$  of degree  $-2 \text{def } \alpha$ . Moreover, if  $\mathbf{s}, \mathbf{t} \in \text{Std}(\lambda)$  and  $\mathbf{u}, \mathbf{v} \in \text{Std}(\mu)$  then*

$$\langle \psi_{\mathbf{st}}, \psi_{\mathbf{uv}}' \rangle_\alpha = \begin{cases} c_{\mathbf{st}}, & \text{if } (\mathbf{u}, \mathbf{v}) = (\mathbf{s}, \mathbf{t}), \\ 0, & \text{if } (\mathbf{u}, \mathbf{v}) \not\sqsubseteq (\mathbf{s}, \mathbf{t}), \end{cases}$$

where  $c_{\mathbf{st}}$  is a non-zero element of  $F$  that depends only on  $\mathbf{s}$  and  $\mathbf{t}$ .

Recall from (1.14) that  $\mathcal{R}_e(\mathfrak{A}_n)_\gamma = (\bigoplus_{\alpha \in \gamma} \mathcal{R}_e(\mathfrak{S}_n)_\alpha)^{\text{sgn}}$ . We will use the bilinear forms on  $\mathcal{R}_e(\mathfrak{S}_n)_\alpha$ , for  $\alpha \in \gamma$ , to define a bilinear form on  $\mathcal{R}_e(\mathfrak{A}_n)_\gamma$ . We do not take the obvious extension of these forms to  $\mathcal{R}_e(\mathfrak{S}_n)_\gamma$ , however, because the arguments below require a **sgn**-invariant form. If  $h \in \mathcal{R}_e(\mathfrak{S}_n)_\gamma$  then  $h = \sum_{\mathbf{i} \in I^\gamma} e(\mathbf{i})h_{\mathbf{i}}$ . Hence, define the trace form  $\tau_\gamma : \mathcal{R}_e(\mathfrak{S}_n)_\gamma \rightarrow F$  by

$$(3.19) \quad \tau_\gamma(e(\mathbf{i})h) = \begin{cases} \tau_\alpha(e(\mathbf{i})h), & \text{if } \mathbf{i} \in I_+^\alpha \subseteq I_+^\gamma, \\ \tau_{\alpha'}(e(-\mathbf{i})h^{\text{sgn}}), & \text{if } \mathbf{i} \in I_-^\alpha \subseteq I_-^\gamma. \end{cases}$$

Importantly,  $\tau_\gamma(h) = \tau_\gamma(h^{\text{sgn}})$ , for all  $h \in \mathcal{R}_e(\mathfrak{S}_n)_\gamma$ . Let  $\langle \cdot, \cdot \rangle_\gamma$  be the corresponding bilinear form on  $\mathcal{R}_e(\mathfrak{S}_n)_\gamma$ .

Extend the dominance ordering  $\supseteq$  to standard tableaux by defining

$$\mathbf{s} \supseteq \mathbf{t} \quad \text{if} \quad \text{Shape}(\mathbf{s}_{\downarrow m}) \supseteq \text{Shape}(\mathbf{t}_{\downarrow m}) \quad \text{for } 1 \leq m \leq n,$$

for  $\mathbf{s}, \mathbf{t} \in \text{Std}(\mathcal{P}_n)$ . We can now prove that  $\mathcal{R}_e(\mathfrak{A}_n)_\gamma$  is a graded symmetric algebra.

**3.20. Theorem.** *Let  $F$  be a field of characteristic different from 2 and suppose that  $e > 2$  and  $\gamma \in Q_n^\varepsilon$ . Then  $\mathcal{R}_e(\mathfrak{A}_n)_\gamma$  is a graded symmetric algebra with homogeneous bilinear form  $\langle \cdot, \cdot \rangle_\gamma$  of degree  $-2 \text{def } \gamma$ .*

*Proof.* By Theorem 3.20 and the definitions above,  $\langle \cdot, \cdot \rangle_\gamma$  is a (not necessarily associative) homogeneous bilinear form on  $\mathcal{R}_e(\mathfrak{S}_n)_\gamma$  of degree  $-2 \text{def } \gamma$ . By restriction, we can consider  $\langle \cdot, \cdot \rangle_\gamma$  as a bilinear form on  $\mathcal{R}_e(\mathfrak{A}_n)_\gamma$ . By construction,  $\langle \cdot, \cdot \rangle_\gamma$  is an associative bilinear form on  $\mathcal{R}_e(\mathfrak{A}_n)$ .

We need to show that  $\langle \cdot, \cdot \rangle_\gamma$  is non-degenerate on  $\mathcal{R}_e(\mathfrak{A}_n)_\gamma$ . To do this we use the two bases of  $\mathcal{R}_e(\mathfrak{A}_n)_\gamma$  given by Corollary 3.15 and Corollary 3.16. Fix  $\lambda, \mu \in \mathcal{P}_n$  and tableaux  $\mathbf{s}, \mathbf{t} \in \text{Std}(\lambda)$ ,  $\mathbf{u}, \mathbf{v} \in \text{Std}(\mu)$  with  $\text{res}(\mathbf{s}) \in I_+^\gamma$  and  $\text{res}(\mathbf{u}) \in I_-^\gamma$ . By

assumption,  $\text{res}(\mathbf{s}) \neq \text{res}(\mathbf{u})$  so  $\langle \psi_{\mathbf{st}}, \psi_{\mathbf{uv}} \rangle_\gamma = \tau_\gamma(\psi_{\mathbf{st}}\psi_{\mathbf{vu}}) = \tau_\gamma(\psi_{\mathbf{vu}}\psi_{\mathbf{st}}) = 0$  by (3.8). Hence,  $\langle \psi_{\mathbf{st}}^{\text{sgn}}, \psi_{\mathbf{uv}}^{\text{sgn}} \rangle_\gamma = \tau_\gamma(\psi_{\mathbf{st}}\psi_{\mathbf{vu}}) = 0$ . Therefore,

$$\begin{aligned} \langle \Psi_{\mathbf{st}}^+, \Psi_{\mathbf{uv}}^+ \rangle_\gamma &= \langle \psi_{\mathbf{st}} + \psi_{\mathbf{st}}^{\text{sgn}}, \psi_{\mathbf{uv}} + \psi_{\mathbf{uv}}^{\text{sgn}} \rangle_\gamma = \langle \psi_{\mathbf{st}}, \psi_{\mathbf{uv}}^{\text{sgn}} \rangle_\gamma + \langle \psi_{\mathbf{st}}^{\text{sgn}}, \psi_{\mathbf{uv}} \rangle_\gamma \\ &= 2\tau_\gamma(\psi_{\mathbf{st}}\psi_{\mathbf{uv}}^{\text{sgn}}) = \begin{cases} \pm 2c_{\mathbf{st}}, & \text{if } (\mathbf{u}', \mathbf{v}') = (\mathbf{s}, \mathbf{t}), \\ 0, & \text{if } (\mathbf{u}', \mathbf{v}') \not\triangleright (\mathbf{s}, \mathbf{t}). \end{cases} \end{aligned}$$

The second last equality follows because  $\tau_\gamma(h) = \tau_\gamma(h^{\text{sgn}})$  for  $h \in \mathcal{R}_e(\mathfrak{S}_n)_\gamma$ , by (3.19), and the last equality follows from Theorem 3.20, since  $\psi_{\mathbf{uv}}^{\text{sgn}} = \pm \psi_{\mathbf{u}'\mathbf{v}'}^{\text{sgn}}$  by Lemma 3.6. Hence, by ordering the two bases  $\{\Psi_{\mathbf{st}}^+\}$  and  $\{\Psi_{\mathbf{uv}}^+\}$  in a way that is compatible with dominance and reverse dominance, respectively, it follows that the Gram matrix  $(\langle \Psi_{\mathbf{st}}^+, \Psi_{\mathbf{uv}}^+ \rangle_\gamma)$  is triangular with non-zero entries on the diagonal. Therefore,  $\langle \cdot, \cdot \rangle_\gamma$  is a non-degenerate associative bilinear form on  $\mathcal{H}_\xi^F(\mathfrak{A}_n)_\gamma$  of degree  $-2 \text{def } \gamma$ , so the theorem is proved.  $\square$

Finally, we describe the blocks and irreducible modules of  $\mathcal{H}_\xi^F(\mathfrak{A}_n)$ .

Let  $\text{Res} = \text{Res}_{\mathcal{R}_e(\mathfrak{A}_n)}^{\mathcal{R}_e(\mathfrak{S}_n)}$  be the restriction functor from the category of finitely generated graded  $\mathcal{R}_e(\mathfrak{S}_n)$ -modules to the category of finitely generated graded  $\mathcal{R}_e(\mathfrak{A}_n)$ -modules.

Let  $\mathcal{R}_n = \{\mu \in \mathcal{P}_n \mid \mu_r - \mu_{r+1} < e \text{ for all } r \geq 1\}$  be the set of  $e$ -restricted partitions of  $n$ . By [2, Theorem 4.11], there is a unique self-dual irreducible graded  $\mathcal{R}_e(\mathfrak{S}_n)$ -module  $D^\mu$  for each  $e$ -restricted partition  $\mu$  and

$$\{D^\mu \langle d \rangle \mid \mu \in \mathcal{R}_n \text{ and } d \in \mathbb{Z}\}$$

is a complete set of pairwise non-isomorphic irreducible graded  $\mathcal{R}_e(\mathfrak{S}_n)$ -modules. By [7, Corollary 5.11], the module  $D^\mu$  arises as a quotient of the corresponding graded Specht module [4, 7].

If  $M$  is an  $\mathcal{R}_e(\mathfrak{S}_n)$ -module let  $M^{\text{sgn}}$  be the  $\mathcal{R}_e(\mathfrak{S}_n)$ -module that is isomorphic to  $M$  as a vector space but where the  $\mathcal{R}_e(\mathfrak{S}_n)$ -action is twisted by  $\text{sgn}$ . By [18, Theorem 3.6.6],  $(D^\mu)^{\text{sgn}} \cong D^{\mathbf{m}(\mu)}$  where  $\mathbf{m}: \mathcal{R}_n \rightarrow \mathcal{R}_n$  is the **Mullineux map**. Therefore, a straightforward application of Clifford theory implies that if  $\mu \neq \mathbf{m}(\mu)$  then  $\text{Res } D^\mu \cong \text{Res } D^{\mathbf{m}(\mu)}$  is an irreducible graded  $\mathcal{R}_e(\mathfrak{A}_n)$ -module and if  $\mu = \mathbf{m}(\mu)$  then, over an algebraically closed field,  $\text{Res } D^\mu = D_+^\mu \oplus D_-^\mu$ , for non-isomorphic irreducible graded  $\mathcal{R}_e(\mathfrak{A}_n)$ -modules  $D_+^\mu$  and  $D_-^\mu$ . Set

$$\mathcal{R}_n^{\mathbf{m}\triangleright} = \{\mu \in \mathcal{R}_n \mid \mathbf{m}(\mu) \triangleright \mu\} \quad \text{and} \quad \mathcal{R}_n^{\mathbf{m}} = \{\mu \in \mathcal{R}_n \mid \mu = \mathbf{m}(\mu)\}.$$

Clifford theory implies that every irreducible graded  $\mathcal{R}_e(\mathfrak{A}_n)$ -module arises in the manner described above, so we obtain the following.

**3.21. Theorem.** *Suppose that  $F$  is an algebraically closed field of characteristic different from 2 and that  $e > 2$ . Then*

$$\{D^\mu \langle d \rangle \mid d \in \mathbb{Z} \text{ and } \mu \in \mathcal{R}_n^{\mathbf{m}\triangleright}\} \cup \{D_+^\mu \langle d \rangle, D_-^\mu \langle d \rangle \mid d \in \mathbb{Z} \text{ and } \mu \in \mathcal{R}_n^{\mathbf{m}}\}.$$

*is a complete set of pairwise non-isomorphic irreducible graded  $\mathcal{R}_e(\mathfrak{A}_n)$ -modules.*

In the semisimple case,  $\mathcal{R}_n = \mathcal{P}_n$  and  $\mathbf{m}(\mu) = \mu'$ . If  $\mu \in \mathcal{P}_n$  and  $\mu = \mu'$  then [19, Proposition 3.9] gives an explicit construction of the modules  $D_+^\mu$  and  $D_-^\mu$  over the field of fractions  $\mathcal{K}$  of the idempotent subring  $\mathcal{O}$  from Definition 2.23.

Finally we turn to the blocks of  $\mathcal{R}_e(\mathfrak{A}_n)$ . By [Corollary 1.15](#),

$$\mathcal{R}_e(\mathfrak{A}_n) = \bigoplus_{\gamma \in Q_n^\varepsilon} \mathcal{R}_e(\mathfrak{A}_n)_\gamma,$$

where  $\mathcal{R}_e(\mathfrak{A}_n)_\gamma$  is a two-sided graded ideal of  $\mathcal{R}_e(\mathfrak{A}_n)$ . Our last result says that if  $\gamma \in Q_n^\varepsilon$  then  $\mathcal{R}_e(\mathfrak{A}_n)_\gamma$  is a block, or indecomposable two-sided ideal, of  $\mathcal{R}_e(\mathfrak{A}_n)$  except when  $\text{def } \gamma = 0$  and  $|\gamma| = 1$ . In the traditional language of the symmetric groups,  $\text{def } \gamma = 0$  if and only if the partitions in  $\mathcal{P}_\gamma$  are  $e$ -cores and, in this case,  $|\gamma| = 1$  if and only if  $\mathcal{P}_\gamma = \{\mu\}$ , where  $\mu = \mathbf{m}(\mu)$  is a Mullineux self-conjugate partition. As  $\mu$  is an  $e$ -core,  $D^\mu$  is an irreducible Specht module and  $\text{Res } D^\mu = D_+^\mu \oplus D_-^\mu$ , similar to the situation considered in the last paragraph.

**3.22. Theorem.** *Suppose that  $F$  is a field of characteristic different from 2 and that  $e > 2$ . Let  $\gamma \in Q_n^\varepsilon$ . Then:*

- a) *If  $|\gamma| = 2$  or  $\text{def } \gamma > 0$  then  $\mathcal{R}_e^F(\mathfrak{A}_n)_\gamma$  is an indecomposable two-sided graded ideal of  $\mathcal{R}_e^F(\mathfrak{A}_n)$ .*
- b) *If  $F$  is algebraically closed,  $|\gamma| = 1$  and  $\text{def } \gamma = 0$  then  $\mathcal{R}_e^F(\mathfrak{A}_n)_\gamma$  is a direct sum of two conjugate matrix algebras.*

*Proof.* This follows by the general theory of covering blocks for  $\mathbb{Z}_2$ -graded algebras as can be found, for example, in [\[24\]](#). In more detail a block  $A$  of  $\mathcal{H}_\xi(\mathfrak{S}_n)$  covers a block  $B$  of  $\mathcal{R}_e(\mathfrak{A}_n)$  if  $B$  is a direct summand of the restriction of  $A$  to  $\mathcal{R}_e(\mathfrak{A}_n)$ . Since  $|\mathbb{Z}_2| = 2$  the blocks of  $\mathcal{R}_e(\mathfrak{A}_n)$  are covered by at most two blocks of  $\mathcal{H}_\xi(\mathfrak{S}_n)$  and, in particular,  $\mathcal{R}_e(\mathfrak{A}_n)_\gamma$  is indecomposable if  $|\gamma| = 2$ . If  $|\gamma| = 1$  and  $\text{def } \gamma > 0$  then there exists a partition  $\mu \in \mathcal{R}_n \cap \mathcal{P}_\gamma$  such that  $\mu \neq \mathbf{m}(\mu) \in \mathcal{P}_\gamma$ . Therefore,  $\mathcal{R}_e(\mathfrak{A}_n)_\gamma$  is a self-conjugate block, so that it is indecomposable. Finally, if  $F$  is algebraically closed,  $\text{def } \gamma = 0$  and  $|\gamma| = 1$  then  $\mathcal{R}_e(\mathfrak{A}_n)_\gamma$  is the direct sum of two matrix algebras, in view of the remarks in the paragraph before the theorem.  $\square$

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